On the Minimal Edge Density of K_4-free 6-critical Graphs

Wenbo Gao and Luke Postle

January 26, 2019

Abstract

Kostochka and Yancey resolved a famous conjecture of Ore on the asymptotic density of k-critical graphs by proving that every k-critical graph G satisfies $|E(G)| \geq (\frac{k}{2} - \frac{1}{k-1})|V(G)| - \frac{k(k-3)}{2(k-1)}$. The graphs for which this bound is tight, k-Ore graphs, contain a large number of K_{k-2}-subgraphs. Subsequent work attempted to determine the asymptotic density for k-critical graphs that do *not* contain large cliques as subgraphs, but only partial progress has been made on this problem. The second author showed that if G is 5-critical and has no K_3-subgraphs, then for $\varepsilon = 1/84$, $|E(G)| \geq (\frac{9}{4} + \varepsilon)|V(G)| - \frac{5}{4}$. It has also been shown that for all $k \geq 33$, there exists $\varepsilon_k > 0$ such that k-critical graphs with no K_{k-2}-subgraphs satisfy $|E(G)| \geq (\frac{k}{2} - \frac{1}{k-1} + \varepsilon_k)|V(G)| - \frac{k(k-3)}{2(k-1)}$. In this work, we develop general structural results that are applicable to resolving the remaining difficult cases $6 \leq k \leq 32$. We apply our results to carefully analyze the structure of 6-critical graphs and use a discharging argument to show that for $\varepsilon_6 = 1/1050$, 6-critical graphs with no K_4 subgraph satisfy $|E(G)| \geq (\frac{k}{2} - \frac{1}{k-1} + \varepsilon_6)|V(G)| - \frac{k(k-3)}{2(k-1)}$.

1 Introduction

A *k-coloring* of a graph G is an assignment $\varphi : V(G) \to \{1, \ldots, k\}$ of one of k colors to each vertex of G, and is a *proper coloring* if $\varphi(u) \neq \varphi(v)$ for every edge uv of G. The *chromatic number* $\chi(G)$ is the smallest integer k for which G has a proper k-coloring. A graph G is said to be *k-critical* if $\chi(G) = k$, and for every proper subgraph H of G, $\chi(H) < k$.

Let $f_k(n)$ denote the minimum number of edges in a k-critical graph with n vertices. It has been a long-standing problem to determine lower bounds on the edge density of k-critical graphs. Early work by Dirac [3] established that k-critical graphs satisfy $|E(G)| \geq \frac{k-1}{2}|V(G)| + \frac{k-3}{2}$, which was subsequently improved in works by Gallai [4], Krivelevich [13], and Kostochka and Steibitz [8, 10], among others. Ore conjectured that $f_k(n+k-1) = f_k(n) + (k-1)(\frac{k}{2} - \frac{1}{k-1})$ [16]. This was finally resolved by Kostochka and Yancey [11]:

Theorem 1.1 (Kostochka and Yancey, Theorem 3 [11])**.** *If G is k-critical, then*

$$|E(G)| \geq \left(\frac{k}{2} - \frac{1}{k-1}\right)|V(G)| - \frac{k(k-3)}{2(k-1)}. \tag{1.1}$$

Furthermore, [11] showed that the *only* graphs for which inequality (1.1) is tight (and which therefore attain minimum edge density) are a special class of graphs known as *k-Ore graphs* (see Definition 2.1). A striking feature is that k-Ore graphs contain many large cliques. It is then natural to ask whether the edge density of k-critical graphs *not* containing large cliques is strictly greater than (1.1). In particular, we are interested in the *asymptotic density* $\lim_n \frac{f_k(n)}{n}$. Ore's conjecture, and Theorem 1.1, imply that the asymptotic density for general k-critical graphs is $\frac{k}{2} - \frac{1}{k-1}$. This leads to a conjecture that the asymptotic density increases when large cliques are excluded.

Conjecture 1.2. *For every $k \geq 4$, there exists $\varepsilon_k > 0$ such that if G is k-critical and does not contain a K_{k-2} subgraph, then*

$$|E(G)| \geq \left(\frac{k}{2} - \frac{1}{k-1} + \varepsilon_k\right)|V(G)| - \frac{k(k-3)}{2(k-1)}.$$

Conjecture 1.2 has been proved for certain values of k. It is vacuously true for $k = 4$, as every 4-critical graph must contain an edge. More is known for $k = 4$; the second author proved in [17] that Conjecture 1.2 holds for 4-critical graphs of girth 5, and Theorem 1.1 was strengthened for such graphs in [15]. The conjecture has also been proved for the case $k = 5$ in [18]. Curiously, the conjecture is also known to be true when k is large; in the thesis [14] (see also [5]), it is shown that Conjecture 1.2 holds for all $k \geq 33$. The case of triangle-free k-critical graphs (for general k) has also been studied in [9]. For $k = 4$, 4-critical graphs with exactly 4 triangles were studied in [1], and near-bipartite 4-critical graphs in [2, 7].

This paper aims to lay the groundwork for resolving the remaining cases $6 \leq k \leq 32$ of Conjecture 1.2, which appear to require more sophisticated and careful analysis. We develop the techniques for this analysis, and then demonstrate them by proving our main theorem, which is the case $k = 6$:

Theorem 1.3. *For $\varepsilon_6 = 1/1050$, 6-critical graphs with no K_4 subgraph satisfy*

$$|E(G)| \geq (\frac{k}{2} - \frac{1}{k-1} + \varepsilon_6)|V(G)| - \frac{k(k-3)}{2(k-1)}.$$

The main elements of this analysis are a modified *potential function* and an operation known as *cloning*, which are used to analyze the local structure of the graph using coarse, global information obtained from measuring the potential. This local information can then be used to obtain global bounds on the number of edges and vertices via *discharging.* To handle the complicated structure of k-critical graphs, we use a combination of *triggered discharging* and *global discharging.*

The *potential function* is a key tool used in [11] to pass between local and global structure. The classical potential, which we denote $\overline{p}(\cdot)$ and refer to as the *KY-potential*,[1] is defined as

$$\overline{p}(G) = (k-2)(k+1)|V(G)| - 2(k-1)|E(G)|.$$

For $R \subseteq V(G)$, the potential of the subgraph induced by R is defined as $\overline{p}(R) = \overline{p}(G[R])$. By rearranging, we immediately observe that Theorem 1.1 is equivalent to the statement that $\overline{p}(G) \leq k(k-3)$ for all k-critical graphs G.

To account for the presence of large cliques, we define

$$T(G) = \max\{2a(H) + b(H) : H \subseteq G \text{ a union of vertex-disjoint cliques}\}$$

where $a(H)$ is the number of components isomorphic to K_{k-1}, and $b(H)$ the number of components isomorphic to K_{k-2}. We now introduce a modification of the potential function. For fixed $\varepsilon > 0, \delta > 0$, the (ε, δ)-*potential* (or simply the *potential*) is defined to be

$$p(G) = ((k-2)(k+1) + \varepsilon)|V(G)| - 2(k-1)|E(G)| - \delta T(G)$$

Note that $p(G) = \overline{p}(G) + \varepsilon|V(G)| - \delta T(G)$. To prove Conjecture 1.2, we make the following restatement:

Conjecture 1.4. *For all $k \geq 6$, there exist $\varepsilon_k, \delta_k, P_k > 0$[2] such that the $(\varepsilon_k, \delta_k)$-potential satisfies*

1. $p(K_k) = k(k-3) + k\varepsilon_k - 2\delta_k$, *and*
2. $p(G) \leq k(k-3) + |V(G)|\varepsilon_k - \left(2 + \frac{|V(G)|-1}{k-1}\right)\delta_k$ *if $G \neq K_k$ is k-Ore, and*
3. $p(G) \leq k(k-3) - P_k$ *if G is k-critical and not k-Ore.*

The non-trivial content of this conjecture is the third statement; the first two statements follow easily from the properties of k-Ore graphs. Rearranging the inequality $p(G) \leq k(k-3) - P_k$ when $T(G) = 0$ yields

$$|E(G)| \geq \left(\frac{k}{2} - \frac{1}{k-1} + \frac{\varepsilon_k}{2(k-1)}\right)|V(G)| - \frac{k(k-3)}{2(k-1)} + \frac{P_k}{2(k-1)}.$$

Thus, Conjecture 1.4 implies Conjecture 1.2, as a k-critical graph G that does not contain a K_{k-2} clique satisfies $T(G) = 0$ and is not k-Ore. To prove our main theorem (Theorem 1.3), we will prove:

Theorem 1.5. *For $k = 6$, Conjecture 1.4 holds for $\varepsilon_6 = \frac{1}{105}, \delta_6 = \frac{10}{105}, P_6 = \frac{20}{21}$.*

[1] to distinguish it from the new (ε, δ)-potential used in our paper, defined below.

[2] Note that ε_k of Conjecture 1.4 is equal to ε_k of Conjecture 1.2 scaled by $2(k-1)$.

1.1 Organization

This paper is organized as follows. In Section 2, we prove several facts about k-Ore graphs and Ore-compositions. In Section 3, we use list colorings to study independent sets of degree $k-1$ vertices. In Section 4, we use the potential function to study the general structural properties of k-critical graphs that are 'close' to violating Conjecture 1.4. In Section 5, we define a notion called *cloning* and develop its properties. The results in Sections 4 and 5 apply to all $k \geq 6$. In Section 6, we apply the general results of the previous sections to 6-critical graphs. In Section 7, we use discharging to prove Theorem 1.5.

1.2 Notation and Conventions

We adopt the following conventions. Unless specifically stated otherwise, a *graph* is a simple graph, and a *coloring* is a proper coloring. For $R \subseteq V(G)$, $G[R]$ denotes the subgraph induced by the vertices in R and $\partial_G R$ denotes the *boundary* of R in G that is $\{v \in R : N(v) \setminus R \neq \emptyset\}$. The edge between vertices u, v will be denoted uv. The set of neighbors of the vertex x is denoted $N(x)$, and the closed neighborhood $\{x\} \cup N(x)$ is denoted $N[x]$. For $R \subseteq V(G)$, y is a neighbor of R if there exists $x \in R$ with $y \in N(x) \setminus R$.

For a set S consisting of pairs $\{u, v\}$ of vertices of G, $G + S$ denotes the graph obtained by adding the edges in S to G. When S consists of a single edge uv, we simply write $G + uv$. For a vertex $v \in V(G)$, $G - v$ denotes the graph obtained by deleting the vertex v. For $A, B \subseteq V(G)$, we use $E_G(A, B)$ to denote the set of edges ab with $a \in A, b \in B$. When G is unambiguous, we simply write $E(A, B)$.

2 k-Ore Graphs and Gems

We first study the properties of k-Ore graphs and a related class of graphs called *gems*, which appear when considering counterexamples to Conjecture 1.4. This immediately yields the first two (easy) statements of Theorem 1.5, and will be useful later. Proofs of the results in this section can be found in [18, 19, 14].

2.1 k-Ore Graphs

Definition 2.1. *An* Ore-composition *of two graphs G_1, G_2 with respect to an edge $xy \in E(G_1)$ and a vertex $z \in V(G_2)$ is the graph obtained by deleting the edge xy, splitting z into two vertices z_1, z_2 of positive degree, and identifying x with z_1 and y with z_2. We refer to G_1 as the* edge-side *of the Ore-composition, G_2 as the* vertex-side*, and z as the* split-vertex*. We refer to x, y as* overlap vertices*, and also write $\overline{xy} = z$ to denote the split-vertex as a vertex of G_2.*

A graph G is k-Ore *if it can be obtained from performing repeated Ore-compositions starting with K_k.*

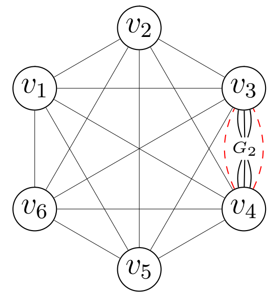

Figure 1: The frame of an Ore-composition.

A graph H obtained from a k-Ore graph G by splitting a vertex v of G into two vertices a, b of positive degree will be called a split k-Ore graph*.*

Recall that a k-Ore graph is obtained by a sequence of Ore-compositions (Definition 2.1) applied to K_k. Suppose G is an Ore-composition of G_1, G_2 on $xy \in E(G_1), z \in V(G_2)$. We can view G as a graph G' isomorphic to G_1, with the edge $xy \in E(G')$ corresponding to the subgraph of G which is isomorphic to G_2 with z split. The edge $xy \in E(G')$ is called a *replacement edge* of the Ore-composition G when viewing G as G'. See Figure 1 for an example where $G_1 = K_k$ and the edge v_3v_4 has been replaced.

When $G_1 = G_2 = K_k$, the Ore-composition of G_1 and G_2 is isomorphic to K_k with a replacement edge. This is true for k-Ore graphs in general, which motivates the following definition.

Definition 2.2. *A* frame *F for the k-Ore graph G is a set of k vertices such that G is isomorphic to the complete graph K_k on F, with some edges possibly being replacement edges.*

Proposition 2.3. *Every k-Ore graph has a frame.*

Given a k-Ore graph H, we use $V^*(H)$ to denote a frame of H, which may be a particular frame if specified, or if not, an arbitrarily chosen frame of H.

Ore-compositions preserve (almost all) cliques, so we can bound the number of large cliques in the resulting graph as the following lemma shows.

Lemma 2.4. *If G is an Ore-composition of G_1, G_2, then $T(G) \geq T(G_1) + T(G_2) - 2$. Moreover, if $G_1 = K_k$ or $G_2 = K_k$, then $T(G) \geq T(G_1) + T(G_2) - 1$.*

Lemma 2.5. *If G is a k-Ore graph and $G \neq K_k$, then $T(G) \geq 2 + \frac{|V(G)|-1}{k-1}$.*

The second statements of Theorem 1.5 and conjecture 1.4 follow from Lemma 2.5.

2.2 Gems

In minimal counterexamples to Conjecture 1.4, certain subgraphs called *gems* arise, which occur if the graph G resembles an Ore-composition.

Definition 2.6. *A subgraph $D \subseteq G$ is a* diamond *if $G[D] = K_k - uv$ and $d_G(x) = k-1$ for every $x \in V(D) \setminus \{u,v\}$. The vertices u, v are called the endpoints of the diamond. A subgraph $D \subseteq G$ is an* emerald *if $G[D] = K_{k-1}$ and $d_G(x) = k-1$ for every $x \in V(D)$. A* gem *is a diamond or emerald.*

Proposition 2.7. *If G is a k-Ore graph and $v \in V(G)$, then there exists a gem of G not containing v.*

Proposition 2.8. *If G is k-Ore and $D = K_{k-1}$ is a subgraph of G, then either $G = K_k$ or there exists a diamond or emerald of G disjoint from D.*

3 Colorings and Independent Sets

In this section, our goal is to obtain an upper bound on the number of vertices of degree k which are adjacent to vertices of degree $k-1$ in a k-critical graph. This will be important when performing global discharging.

We introduce several definitions which are needed for this section. Given a function $L(v)$ which associates a list of colors to each vertex v, an assignment $\varphi : V(G) \to \mathbb{N}$ is an *L-coloring* if $\varphi(v) \in L(v)$ for all v, and $\varphi(u) \neq \varphi(v)$ for every edge uv. Given a function $f : V(G) \to \mathbb{N}$, the graph G is *f-choosable* if an L-coloring exists for every function $L(v)$ with $|L(v)| = f(v)$ for all v.

In our discharging argument (Section 7.4), it is necessary to bound the number of edges emanating from independent sets consisting of degree $k-1$ vertices. To do so, we employ the following result on choosability [6]:

Lemma 3.1 (Kierstead and Rabern, Main Lemma [6])**.** *Let G be a non-empty graph and let $f : V(G) \to \mathbb{N}$ with $f(v) \leq d_G(v) + 1$ for all $v \in V(G)$. If there is an independent set $A \subseteq V(G)$ such that*

$$|E(A, V(G))| \geq \sum_{v \in V(G)} d_G(v) + 1 - f(v)$$

then G has a non-empty induced subgraph H that is f_H-choosable, for $f_H(v) = f(v) + d_H(v) - d_G(v)$.

Our interest is the case where A consists of vertices of degree $k-1$.

Lemma 3.2. *Let G be a k-critical graph, and let $A \subseteq V(G)$ be an independent set with $d_G(v) = k-1$ for all $v \in A$. Let B_i be the set of vertices v that have a neighbor in A such that $d_G(v) = k-1+i$. Then for any $\ell \geq 1$, defining $B = \bigcup_{i=1}^{\ell} B_i$, we have*

$$|E_G(A, B)| \leq |A| + \sum_{i=1}^{\ell} (i+1)|B_i|.$$

Proof. See Appendix A.1. □

In Section 7, we will apply the following special case of Lemma 3.2.

Corollary 3.3. *Let G be a k-critical graph, and let $A \subseteq V(G)$ be an independent set with $d_G(v) = k-1$ for all $v \in A$. Let B be the set of neighbors of A of degree k. Then $|E_G(A,B)| \leq |A| + 2|B|$.*

4 Applications of the Potential Function

Recall the definition of the (ε, δ)-potential function as follows.

Definition 4.1. *The* potential *of a graph G is given by*

$$p(G) = ((k-2)(k+1)+\varepsilon)|V(G)| - 2(k-1)|E(G)| - \delta T(G)$$

For $R \subseteq V(G)$, the potential $p_G(R)$ is $p(G[R])$.

This section is devoted to analyzing the local structure of k-critical graphs for $k \geq 6$, by measuring the contribution of edges and vertices to the potential in substructures of the graph. The key result is that, broadly speaking, k-critical graphs with high potential cannot contain any 'nearly' k-critical graphs as substructures, which can be made k-critical by adding a small number of edges. To make this precise, we state a few definitions.

Definition 4.2. *We say a graph H is* smaller *than G if any of the following hold:*

1. *$|V(H)| < |V(G)|$, or*
2. *$|V(H)| = |V(G)|$ and $|E(H)| < |E(G)|$, or*
3. *$|V(H)| = |V(G)|$, $|E(H)| = |E(G)|$, and H has more pairs of vertices with the same closed neighborhood than G.*

We say a k-critical graph G is good *if every graph smaller than G satisfies Conjecture 1.4. We say G is i-*tight *if G is good and $p(G) > k(k-3) - P - Q + i\delta$ for an auxiliary parameter Q (see Section 4.1), and we write* tight *for 0-tight.*

Tight graphs are k-critical graphs which are 'close' to being counterexamples to Conjecture 1.4.

Since the results of this section generalize those in the preceding works [18, 14, 11], we defer the proofs to Appendix A.2.

Definition 4.3. *An* edge-addition *in G is a non-edge xy such that $G + xy$ contains a k-critical subgraph H with $V(H) \subsetneq V(G)$.*

*An i-*edge-addition *in G is a set S of at most i non-edges such that $G + S$ contains a k-critical subgraph H with $V(H) \subsetneq V(G)$.*

We will show (as a part of the more general Lemma 4.17) the following.

Proposition 4.4. *Let G be a tight, ungemmed graph. Then for $1 \leq i \leq (k-2)/2$, G does not admit an i-edge-addition.*

The next results demonstrate how the nonexistence of edge additions can be used to deduce the local structure of the graph. The following lemmas will be used frequently in subsequent sections.

Lemma 4.5. *Suppose that G does not admit an i-edge-addition for some $i \geq 1$. Let x be a vertex of degree $k-1$ and y a neighbor of x of degree at most $k-2+i$. If $|N(x) \cap N(y)| \geq k-3$, then $N[x] \subseteq N[y]$.*

Corollary 4.6. *Suppose G does not admit edge-additions. If x, y are adjacent vertices of degree $k-1$ with $|N(x) \cap N(y)| \geq k-3$, then $N[x] = N[y]$.*

Corollary 4.7. *Suppose G does not admit 2-edge-additions. If x, y are adjacent, with $d(x) = k-1, d(y) = k$, and $|N(x) \cap N(y)| \geq k-3$, then $N[x] \subseteq N[y]$.*

4.1 Assumptions

Let $k \geq 6$ be fixed. We subsequently write ε, δ, P for the constants $\varepsilon_k, \delta_k, P_k$ in Conjecture 1.4. Unless stated otherwise, G is always k-critical.

In this section and following sections, we will assume that ε, δ, P, and auxiliary constants Q, Δ, Γ, satisfy the conditions of the forthcoming Assumption 1. These inequalities may not be sufficient to determine precise values of $\varepsilon_k, \delta_k, P_k$ for which Conjecture 1.4 holds for a specific k, as additional restrictions on ε_k may arise during the discharging step. However, so long as the following inequalities are satisfied, the results of Sections 4 and 5 will hold for any $k \geq 6$.

Assumption 1 All of the following inequalities hold:

1. $\delta = 2(k-1)\varepsilon$, and
2. $2 + \delta \leq \Gamma \leq k-2$, and
3. $(\Gamma - 2) + Q + k\delta \leq P$, and
4. $P + Q + \frac{k}{2}\delta \leq \Delta$, and
5. $\Delta + (k-1)\delta \leq 2$.

For the asymptotic density, we are interested in maximizing the value of ε. In the non-asymptotic case, we may also be interested in maximizing P. We state our results in this section in terms of these constants (and not ε only) both for increased clarity, and for this flexibility.

4.2 Analysis of Potential

This section extends the theory of the KY-potential $\overline{p}$ to the (ϵ, δ)-potential.

Definition 4.8. *Suppose G is a k-critical graph, and $R \subsetneq V(G)$. By definition, there exists a $(k-1)$-coloring $\varphi : R \to \{1, \ldots, k-1\}$ of $G[R]$. We define $G_{R,\varphi}$ to be the graph obtained by identifying all vertices of the same color in φ to a single vertex x_i for all $i \in \{1, \ldots, k-1\}$, adding the edges $x_i x_j$ for all $i \neq j \in \{1, \ldots, k-1\}$, and removing multiple edges[3].*

[3]To obtain a simple graph.

Note that $G_{R,\varphi}$ is not $(k-1)$-colorable, or else G would have been $(k-1)$-colorable. Thus $G_{R,\varphi}$ contains a k-critical subgraph, which necessarily includes at least one of the vertices $x_1, \dots, x_{k-1}$. Let W be such a subgraph. Let $X = V(W) \cap \{x_1, \dots, x_{k-1}\}$ and $R' = R \cup (V(W) - X)$. Note that $R \subsetneq R' \subseteq V(G)$. We say R' is a critical extension *of R with* extender *W and* core *X.*

Definition 4.9. *Let R' be a critical extension of $R \subsetneq V(G)$ with extender W and core X. If $R' = V(G)$, we say R' is* spanning. *If $|E(G[R'])| = |E(G[R])| + |E(W)| - |E(K_X)| + i$, we say R' is* i-incomplete. *R' is* complete *if $i = 0$.*

Lemma 4.10. *If R' is an i-incomplete critical extension of $R \subsetneq V(G)$ with extender W and core X, then*

$$p_G(R') \leq p_G(R) + p(W) - 2i(k-1) - (p(K_X) + \delta T(K_X) - \delta|X|).$$

A key property is that the (ϵ, δ)-potential decreases under critical extensions. Lemma 4.11 implies that every proper subgraph $G[R]$ has higher potential than G, as the potential decreases by iterating critical extensions.

Lemma 4.11. *Let G be good, and $R \subsetneq V(G)$ with $G[R]$ not a clique. If R' is an i-incomplete critical extension of R, then $p_G(R') \leq p_G(R) - 2(i+1)(k-1) - \delta$.*

4.3 Collapsible Subsets and Edge Additions

The following definition is useful for characterizing the proper subsets of G which have the lowest potential.

Definition 4.12. *A subset $R \subsetneq V(G), |R| \geq 2$ is* collapsible *in G if for every proper $(k-1)$-coloring of $G[R]$, the vertices of $\partial_G R$ receive the same color.*

A subset $R \subsetneq V(G), |R| \geq 2$ is i-collapsible[4] *in G if for all $(k-1)$-colorings $\varphi : R \to C = \{1, \dots, k-1\}$ of $G[R]$,*

$$\min_{1 \leq c \leq k-1} |E(\varphi^{-1}(C \setminus \{c\}) \cap R, V(G) \setminus R)| \leq i.$$

We will see that i-collapsible subsets correspond to $(i+1)$-edge additions for $i \leq (k-3)/2$. We begin by providing a characterization of collapsible subsets.

Proposition 4.13. *A subset $R \subsetneq V(G), |R| \geq 2$ is collapsible in G if and only if every critical extension has a core of size 1, is spanning, and is complete.*

Proposition 4.13 has a partial converse for i-collapsible subsets.

Lemma 4.14. *If $R \subsetneq V(G)$ is such that all critical extensions of R are spanning, have a core of size 1, and are at most i-incomplete, then R is i-collapsible.*

The following lemma shows that such a critical subgraph arising from an i-edge-addition yields a subset of $V(G)$ whose potential is small.

[4] It is easy to see that collapsibility corresponds to 0-collapsibility.

Lemma 4.15. *Suppose G is tight and admits an i-edge-addition. Let H be a k-critical subgraph of $G+S$ and let $R = V(H)$. Then*

$$p_G(R) \leq p(G) + P + Q + 2i(k-1) + i\delta.$$

Furthermore, if H is not k-Ore, then $p_G(R) \leq p(G) + Q + 2i(k-1) + i\delta$.

Lemma 4.16. *If G contains an i-collapsible subset R for $i \leq (k-3)/2$, then G admits an $(i+1)$-edge-addition S consisting of non-edges of $G[R]$.*

Now we are ready to prove the key lemma which shows that G does not admit i-edge-additions for $i \leq (k-2)/2$, and in turn provides a lower bound on the potential of proper subsets of G. This strengthens a similar result from [14, 5], which obtains the nonexistence of i-edge additions only up to $(k-4)/2$; eliminating one more edge-addition turns out to be very powerful for small k.

Lemma 4.17. *Let G be a tight, ungemmed graph. Then for all i with $1 \leq i \leq (k-2)/2$, G does not admit an i-edge-addition, and there is no subset $R \subsetneq V(G)$, $G[R]$ not a clique, with $p_G(R) < p(G) + 2i(k-1) + \Delta$.*

5 Cloning

In this section, we introduce an operation known as *cloning* and use it to study the structure of tight, ungemmed graphs. Applied to vertices of low degree, cloning produces subgraphs that resemble K_k, which we refer to as *clusters*. Using the potential function and the results of Section 4, we can infer facts about the structure of the cluster, and therefore of the original graph, which are of great importance when we later perform discharging.

We begin by defining cloning and clusters, and analyzing them. A key result is that a tight, ungemmed graph cannot have a cluster of size greater than $k-4$. We then turn our attention to a particular subgraph called a *gadget* which arises from cloning, and has a recursive structure that allows for carefully targeted discharging. Finally, we obtain several results about minimal counterexamples.

5.1 Cloning and Clusters

In this section, we define an operation called *cloning*, which copies a vertex of degree $k-1$ and deletes one of its neighbors. We then use it to derive structural properties of a minimum counterexample. Unless stated otherwise, G is k-critical. The constants $\varepsilon, \delta, P, Q, \Delta, \Gamma$ satisfy Assumption 1. Similar results on cloning and clusters have appeared in [18, 19, 14], so we defer the proofs to Appendix A.3.

Definition 5.1. *Let x, y be vertices with $d(x) = k-1$ and $xy \in E(G)$. The graph $G_{y\to x}$ is obtained by letting $V(G_{y\to x}) = V(G) \setminus \{y\} \cup \{\widetilde{x}\}$ and $E(G_{y\to x}) = E(G-y) \cup \{\widetilde{x}v \mid v \in N_G(x)\} \cup \{\widetilde{x}x\}$. This operation is called* cloning *x with y.*

Definition 5.2. *A* cluster *is a maximal set $R \subseteq V(G)$ such that $d(x) = k-1$ for every $x \in R$, and $N[x] = N[y]$ for every $x, y \in R$.*

For a cluster C, $G_{y \to C}$ denotes the cloning of a vertex of C with y. It is easy to see that this is independent of the vertex of C chosen. When we use the notation $G_{y \to C}$, we use $\widetilde{y}$ to denote the new vertex of the clone (note this is the opposite of the notation $\widetilde{x}$ used when we clone $G_{y \to x}$).

Lemma 5.3. *If x and y are vertices such that x is in a cluster C of size s, $xy \in E(G)$, and $d(y) \leq k-2+s$, then $G_{y \to x}$ is not $(k-1)$-colorable.*

Lemma 5.4. *Suppose G is a tight, ungemmed graph, and $xy \in E(G)$ such that*

1. *x is in a cluster C_x of size s*
2. *$d(y) \leq k-2+s$*
3. *If y is in a cluster C_y, then $C_y \neq C_x, |C_y| \leq s$.*

If $H \subseteq G_{y \to x}$ is k-critical, then either H is k-Ore, or, $H = G_{y \to x}$ and $d_G(y) = k-1$.

Lemma 5.5. *Let G be a tight, ungemmed graph. If $R \subsetneq V(G), |R| \geq k$ is a proper subset, then $p_G(R) > p(G) + (k-1)(k-2) + (\Gamma + Q + \delta)$ unless $G - R$ is a single vertex of degree at most $k - 1 + (k-4)/2$ in G.*

Next, we show that a cluster in a tight, ungemmed graph has size at most $k-4$. This is a crucial tool when used in conjunction with cloning, and is vital for analyzing the structure of hypothetical counterexamples to Conjecture 1.2. For comparison, a similar result [11, Claim 20] excludes clusters larger than $k-3$ (in the KY-potential setting). The proof of Lemma 5.6 makes use of a stronger result by Kostochka and Yancey [12, Theorem 6] which extends Theorem 1.1.

Lemma 5.6. *If G is tight and ungemmed, then a cluster has size at most $k-4$.*

We also have the following facts about cloned graphs.

Lemma 5.7. *If G is a tight, ungemmed graph, x is a vertex of degree $k-1$, and y is a neighbor of x, then $G_{y \to x}$ is ungemmed.*

Lemma 5.8. *If C is a cluster of size s, x is a vertex of C, and y is a neighbor of x with degree at most $k-2+s$, then $p(G_{y \to x}) \geq p(G) - \delta$.*

5.2 Gadgets

Several structures resembling k-Ore subgraphs arise from cloning, and their properties will be crucial for analyzing the discharging. Suppose that G is tight and ungemmed, and x, y are vertices satisfying the conditions of Lemma 5.4, such that $G_{y \to x}$ contains a k-Ore subgraph H. We first show that H has a unique frame which contains $x, \widetilde{x}$.

Lemma 5.9. *Let G be tight and ungemmed, and suppose that*

1. *x, y satisfy the conditions of Lemma 5.4*
2. *$G_{y \to x}$ contains a k-Ore subgraph H*

Then H has a unique frame $V^*(H) = (N(x) \setminus \{y\}) \cup \{\widetilde{x}\}$.

This lemma shows that the clone $G_{y \to x}$ has a K_k-subgraph with replacement edges, and with $x, \widetilde{x}$ vertices of K_k. Then upon deleting $\widetilde{x}$, G contains a subgraph that can be viewed as K_{k-1} with replacement edges, with x a vertex of K_{k-1}. This motivates the following definitions.

Definition 5.10. *A* gadget *is a subgraph H obtained from a k-Ore graph by deleting a vertex x of degree $k-1$ in a cluster of size at least 2.*

A proto-gadget *is a subgraph H isomorphic to K_{k-1} with replacement edges, along with a distinguished frame $V^*(H) = V(K_{k-1})$.*

A kite *is a subgraph H isomorphic to K_{k-2} with replacement edges, along with a distinguished frame $V^*(H) = V(K_{k-2})$.*

Lemma 5.11. *Let G be tight and ungemmed, and let H be a split k-Ore subgraph of G, with overlap vertices $\{a, b\}$. If $x \in V(H) \setminus \{a, b\}$, then $x \in V^*(H')$ for some proto-gadget H' of G with $H' \subseteq H - \{a, b\}$.*

Recall that a replacement edge e between the vertices a, b is precisely a split k-Ore subgraph H with overlap vertices $\{a, b\}$. Abstractly, we wish to be able to treat the graph G as containing an edge ab instead of the subgraph H, and more generally, given a recursive description of the graph in terms of successive replacement edges, we should be able to treat higher-level representations of the graph without reference to the internal structure of the replacements. This motivates the next definition.

Definition 5.12. *Given a replacement edge e corresponding to a split k-Ore subgraph H with overlap vertices $\{a, b\}$, we say that a set of vertices S is* inside (the replacement edge) *e if $S \subseteq V(H) \setminus \{a, b\}$.*

In this language, Lemma 5.11 shows that if e is a replacement edge and x is inside e, then there is an entire proto-gadget H' inside e with $x \in V(H')$.

5.3 Structural Properties of a Counterexample

The following are applications of cloning. We defer the proofs to Appendix A.3.

Lemma 5.13. *If G is a tight, ungemmed graph, then G does not contain an induced path of length 3 consisting of vertices of degree $k-1$.*

Lemma 5.14. *A minimal counterexample to Conjecture 1.4 is 3-connected.*

Lemma 5.15. *A minimal counterexample to Conjecture 1.4 does not admit an edge-addition.*

6 6-Critical Graphs

We now prove Theorem 1.5. The proof is divided into two parts. In this section, we first analyze the local, structural properties of 6-critical graphs by applying the results of Section 4 and Section 5. We then combine this with discharging in Section 7 to complete the proof. Let

$$\varepsilon = \frac{1}{105},\ \delta = 10\varepsilon = \frac{10}{105},\ P = \frac{20}{21},\ Q = \frac{2}{7},\ \Delta = \frac{32}{21},\ \Gamma = 2 + \frac{10}{105}$$

One can verify by routine calculations that these values satisfy Assumption 1 (see Section 4.1), and that the inequalities arising in the discharging analysis of Section 7 are satisfied for $\varepsilon = \frac{1}{105}$.

Lemma 6.1. *If G is a minimal counterexample to Theorem 1.5, then G is 3-tight and ungemmed.*

Proof. Suppose not. Since $p(G) > k(k-3) - P$ and $Q \geq 3\delta$, G is 3-tight. By Lemma 5.15, G does not admit an edge-addition and hence G does not contain a diamond D. If G contains an emerald D, then by Lemma 5.15 and Corollary 4.6, the vertices of D have a common neighbor v not in D. But then $G[V(D) \cup \{v\}] = K_k$, a contradiction since G is not k-Ore. Hence G is 3-tight and ungemmed, a contradiction. □

In Section 7, we will assign charge to $V(G)$ in such a way that vertices of degree 5 have negative initial charge, and all other vertices have positive initial charge. Our goal is to discharge the vertices so that every vertex has non-negative charge, which will produce a contradiction. To that end, we first study the structural properties of degree 5 vertices.

Definition 6.2. *For each $i \geq 5$, $D_i(G)$ denotes the subgraph of G induced by vertices of degree i.*

Lemma 6.3. *If $H \subseteq G$ is a split k-Ore subgraph with $V(H) \subsetneq V(G)$ and overlap vertices $\{a, b\}$, then $d_H(a), d_H(b) \geq 3$.*

Proof. Suppose not. We may assume without loss of generality that $d_H(a) \leq 2$. Let $G' = G[V(H)] + \{bu : u \in N_H(a)\}$. Since $N_{G'}(b)$ contains $N_H(a) \cup N_H(b)$, it follows that G' contains a 6-Ore subgraph and hence G admits a 2-edge-addition, contradicting by Lemma 4.17. □

We will frequently make use of Lemma 6.3. In particular, Lemma 6.3 yields:

Corollary 6.4. *Let $x \in V^*(H)$ for a gadget H of G. If $d_G(x) = 5$, then x is not incident to a replacement edge of H, and if $d_G(x) \in \{6, 7\}$, then x is incident to at most 1 replacement edge of H.*

Proof. It follows from Lemma 6.3 that $d_G(x) \geq 4 + 2r$ where r is the number of replacement edges of H that x is incident to. Hence if $d_G(x) = 5$, then $r = 0$ and if $d_G(x) \leq 7$, then $r \leq 1$. □

Corollary 6.5. *Let H be a proto-gadget of G. If $x, y \in V^*(H)$ and $d_G(x) = d_G(y) = 5$, then x and y are in the same cluster of G.*

Proof. By Corollary 6.4, neither x nor y are incident to a replacement edge of H. Hence x and y are adjacent and $|N(x) \cap N(y)| \geq 3$. By Corollary 4.6, x and y are in the same cluster of G as desired. □

Lemma 6.6. *If G is a 1-tight, ungemmed graph and C is a cluster of size 2 in G, then C has at most 1 neighbor having degree at most 6. Furthermore, if C has a neighbor of degree at most 6, then C is contained in a proto-gadget of G.*

Proof. Let $\{v_1, v_2, v_3, v_4\}$ denote the neighbors of C. Suppose C has a neighbor of degree at most 6, which we assume to be v_1. By Lemma 5.4, $G_{v_1 \to C}$ is 6-critical or $G_{v_1 \to C}$ contains a 6-Ore subgraph. Lemma 5.8 implies that $G_{v \to C}$ is tight and Lemma 5.7 implies that $G_{v \to C}$ is ungemmed. But $G_{v \to C}$ has a cluster of size 3, and therefore is not 6-critical, or else it would contradict Lemma 5.6. Hence $G_{v \to C}$ contains a 6-Ore subgraph H_1. By Lemma 5.9, $V^*(H_1) = C \cup \{\widetilde{v_1}, v_2, v_3, v_4\}$, and $H_1 - \widetilde{v_1}$ is a proto-gadget of G containing C. This proves the second statement.

To prove the first statement, let us further suppose for contradiction that C has a second neighbor (wlog, v_2) having degree at most 6. Since v_2 has degree at most 7 in $G_{v_1 \to C}$, Lemma 6.3 implies that v_2 is adjacent to at least one of v_3 and v_4. Without loss of generality, we may assume that v_2 is adjacent to v_3.

If $d(v_2) = 5$, then v_2 has degree 6 in $G_{v_1 \to C}$ and Corollary 6.4 implies that v_2 is adjacent to both v_3, v_4. But then Corollary 4.6 implies that v_2 is in the cluster C, a contradiction. So we may assume $d(v_2) = 6$. Now consider $G_{v_2 \to C}$. By the same reasoning as above, $G_{v_2 \to C}$ contains a 6-Ore subgraph H_2 with $V^*(H_2) = C \cup \{\widetilde{v_2}, v_1, v_3, v_4\}$. Using the replacement edges of H_2, we find that $G[(V(H_2) \setminus \{\widetilde{v_2}\}) \cup \{v_2\}\}] + v_2v_1 + v_2v_4$ contains a 6-Ore subgraph. If $(V(H_2) \setminus \{\widetilde{v_2}\}) \cup \{v_2\} \subsetneq V(G)$, then G admits a 2-edge-addition, contradicting Lemma 4.17. So we may assume that $(V(H_2) \setminus \{\widetilde{v_2}\}) \cup \{v_2\} = V(G)$. But now $|V(G)| = |V(H_2)|$ and $|E(G)| \geq |E(H_2)| + 1$. Since H_2 is 6-Ore, we find that the potential of G is at most $p(G) \leq p(H_2) - 2(k-1) + \delta < k(k-3) - P - Q$, which contradicts that G is 1-tight. □

Applying Corollary 6.4 and lemma 6.6 yields the nonexistence of many structures in 2-tight, ungemmed graphs. We defer the proofs to Appendix A.4.

Lemma 6.7. *If G is a 2-tight, ungemmed graph, then $D_5(G)$ does not contain a subgraph isomorphic to K_4.*

Lemma 6.8. *Let G be a 2-tight, ungemmed graph. If $P = v_1v_2v_3$ is an induced path such that $d_G(v_1) = d_G(v_2) = d_G(v_3) = 5$, then $G_{v_1 \to v_2}$ and $G_{v_3 \to v_2}$ are 1-tight and ungemmed.*

Lemma 6.9. *Let G be a 2-tight, ungemmed graph. If a component of $D_5(G)$ contains a triangle, then that component is a triangle.*

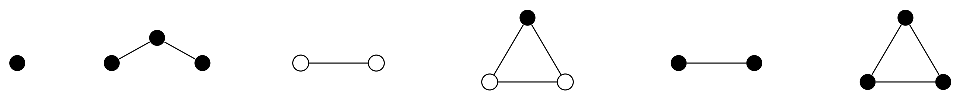

Figure 2: Components of $D_5(G)$. Open circles represent vertices in the same cluster.

Lemma 6.10. *If G is a 2-tight, ungemmed graph, then $D_5(G)$ does not contain a 4-cycle.*

Lemma 6.11. *If G is a 2-tight, ungemmed graph, then the maximum degree of $D_5(G)$ is at most 2.*

Corollary 6.12. *Let G be a 2-tight, ungemmed graph. Every component of $D_5(G)$ has size at most 3.*

Proof. From Lemma 5.13, Lemma 6.9, Lemma 6.10, and Lemma 6.11, we find that no component of $D_5(G)$ can have more than 3 vertices. □

The possible components in $D_5(G)$ are shown in Figure 2.

Lemma 6.13. *Let G be a 2-tight and ungemmed graph, and let $H \subseteq G$ be a proto-gadget. If $x \in V^*(H)$ has degree 5 in G, and $y \notin V^*(H)$ is a neighbor of x having degree at most 7 in G, then there does not exist a proto-gadget H' with $x, y \in V^*(H')$.*

Proof. Suppose not. That is, there exists a proto-gadget H' with $x, y \in V^*(H')$. Since x has degree 5 in G, by Corollary 6.4, x is not incident with a replacement edge of H or H'. Let $\{z_1, \ldots, z_4\}$ denote the neighbors of x in $V^*(H)$. Since $x \in V^*(H')$, exactly one of $z_1, \ldots, z_4$ is not in $V^*(H')$; we may assume without loss of generality that $z_1 \notin V^*(H')$.

First suppose y has degree 5 in G. Then by Corollary 6.4, y is not incident with a replacement edge of H'. Hence y is adjacent to all of z_2, z_3, and z_4. By Corollary 4.6, x and y are in the same cluster. But then $G' = G[H \cup \{y\}]$ is a k-Ore subgraph of H. Since G is 6-critical, we find that $G = G'$. By Proposition 2.7, it follows that G contains a gem, a contradiction.

So we may assume that y has degree 6 or 7 in G. Since y has degree at most 7 in G, by Corollary 6.4 y is incident with at most 1 replacement edge in H'. Thus, y is adjacent to at least two of the vertices $\{z_2, z_3, z_4\}$. We may assume without loss of generality that y is adjacent to z_3 and z_4. Adding the edges yz_1, yz_2 to the proto-gadget H yields a 6-Ore subgraph K of G.

We claim that $V(K) = V(H) \cup \{y\}$ is a proper subset of $V(G)$. Let $S = E(G) \setminus E(K)$. Since y has degree 6 or 7, $|S| \geq 3$. Therefore, since $E(K) = E(H) \cup \{yx, yz_1, yz_2, yz_3, yz_4\}$, we have that $|E(G)| \geq |E(K)| + 1$. If $|V(G)| = |V(K)|$, then $p(G) \leq p(K) - 2(k-1) + \delta < k(k-3) - P - Q$, which contradicts that G is 2-tight. Therefore $V(K) \subsetneq V(G)$. But then G admits a 2-edge-addition, contradicting Lemma 4.17. □

Corollary 6.14. *Let G be a 2-tight and ungemmed graph, and let $H \subseteq G$ be a proto-gadget. Suppose $x \in V^*(H)$ has $d_G(x) = 5$ and y is a neighbor of x having degree at most 7. If $y \notin V^*(H)$, then y is the* unique *neighbor of x with the property that there is no proto-gadget H' with $x, y \in V^*(H')$.*

In Section 7, we will see that proto-gadgets have large initial charge. Vertices v of degree 6 or 7 incident to a replacement edge e should avoid sending charge to neighbors inside e, as those neighbors can receive sufficient charge from proto-gadgets inside e. This allows us to distribute the charge from v more efficiently. To make this precise, we introduce the following notion.

Definition 6.15. *Let x be a vertex of degree 5, and y a neighbor of x having degree 6 or 7. We say x is a* downward neighbor *of y if there exists a proto-gadget H with $x \in V^*(H), y \notin V^*(H)$. If x is not a downward neighbor of y, then we say that x is an* upward neighbor *of y.*

Similarly if C is a cluster and y is a neighbor of C having degree 6 or 7, then we say y is a downward (resp. upward) neighbor *of C if y is a downward (resp. upward) neighbor of every vertex $v \in C$.*

Note that for the last definition with clusters, a neighbor is either downward or upward. This follows since if y is downward of some vertex $v \in C$, then y is downward of every vertex in C. To see this, note that there exists a proto-gadget H with $v \in V^*(H), y \notin V^*(H)$, but then $C \setminus \{v\} \subseteq N(v) \setminus \{y\} \subseteq V^*(H)$ by Corollary 6.4 as v has degree 5 in G.

Corollary 6.16. *Suppose $H \subseteq G$ is a split k-Ore subgraph with overlap vertices $\{y, z\}$. If y has degree 6 or 7 and is adjacent to a vertex $x \in V(H)$ having degree 5, then x is a downward neighbor of y.*

Proof. By Lemma 5.11, there exists a proto-gadget $H' \subseteq H - \{y, z\}$ with $x \in V^*(H')$. Clearly $y \notin V(H')$, so x is a downward neighbor of y. □

Corollary 6.17. *If $H \subseteq G$ is a proto-gadget with $x, z \in V^*(H)$ such that x has degree 5 in G and z has degree 6 or 7, then x is an upward neighbor of z.*

Proof. Since $d_G(x) = 5$, by Corollary 6.4, x is not incident with a replacement edge of H. Hence there exists a unique neighbor y of x with $y \notin V^*(H)$. By Corollary 6.14, y is the unique neighbor of x such that there does not exist a proto-gadget H' with $x, y \in V^*(H')$. If x were a downward neighbor of z, then there would exist a proto-gadget H' with $x \in V^*(H'), z \notin V^*(H')$. Since $d_G(x) = 5$ and $d_{H'}(x) \geq 4$, we find that $y \in V^*(H')$, a contradiction. □

7 Discharging

Suppose that G is a minimal counterexample to Theorem 1.5. By Lemma 6.1, G is 3-tight and ungemmed. Recall that $\varepsilon = \frac{1}{105}$; in this section, ε is the only constant which appears, and the reader can verify that the numerical inequalities are satisfied by $\varepsilon = \frac{1}{105}$.

Definition 7.1. *The initial charge* $ch_0 : V(G) \to \mathbb{R}$ *is*

$$ch_0(v) = (k-1)d(v) - (k-2)(k+1) - \varepsilon.$$

In particular, when $k = 6$*, the initial charge is* $5d(v) - 28 - \varepsilon$*, which is equal to* $-3 - \varepsilon, 2 - \varepsilon, 7 - \varepsilon, 12 - \varepsilon, \ldots,$ *etc. resp. for vertices of degree* $5, 6, 7, 8, \ldots,$ *etc.*

The total charge of the vertices of the graph is

$$\sum_{v \in V(G)} ch_0(v) = 2(k-1)|E(G)| - ((k-2)(k+1) + \varepsilon)|V(G)| = -p(G) - \delta T(G).$$

As G is a counterexample to Theorem 1.5, $-p(G) - \delta T(G) < -(k(k-3) - P) - \delta T(G) < 0$. Our objective is to transfer charge to the vertices of degree 5, which have negative initial charge, until every vertex has non-negative charge, yielding a contradiction. We will move charge between vertices in six stages, each composed of discharging rules. The charge after the termination of STAGE i will be denoted $ch_i(v)$. Several of the stages involve successive rounds of discharging, which 'trigger' when conditions are met.

Definition 7.2. *For a subset* $S \subseteq V(G)$*, define* $ch_i(S) = \sum_{v \in S} ch_i(v)$*. A cluster* C *is* satisfied *if* $ch_i(C) \geq 0$*, and* happy *if* $ch_i(C) \geq 2 + 2\varepsilon$*. We say* C *is* unsatisfied *(resp.* unhappy*) if it is not satisfied (resp. happy). A vertex* v *in the cluster* C *is satisfied (resp. happy) if* C *is satisfied (resp. happy). A component* S *of* $D_5(G)$ *is* satisfied *if* $ch_i(S) \geq 0$.

The discharging process is quite involved. For clarity, we first provide a high-level overview of the rules and objectives of each stage.

1. In the first two stages (Section 7.1), vertices with a large amount of excess positive charge send their charge to degree 5 neighbors. In STAGE 1, vertices of degree at least 8 send charge equally to their degree 5 neighbors.

   In STAGE 2, vertices of degree 6 and 7 release charge to upward neighbors only when certain rules are triggered. STAGE 2 ends when the trigger condition is not met by any vertex. After STAGE 2, degree-5 vertices which are in gadgets, or have high-degree neighbors, are happy.

2. In STAGE 3 (Section 7.2), we discharge the remaining degree-6 and degree-7 vertices. All undischarged vertices of degree 7 release their charge, and the remaining vertices of degree 6 discharge when a specific condition is met. Vertices of degree 6 which have not yet discharged by the end of STAGE 3 will be called *reserved vertices*.

   Reserved vertices have a strict structural characterization. The degree-5 vertices which are not satisfied after STAGE 3 must either be adjacent to a large number of reserved vertices, or else also have a specific structural characterization (which we call *dangling* vertices).

   In Section 7.3, we analyze the result of the previous stages to determine which vertices still require charge after STAGE 3. Clusters of size 2 and induced paths of length 2 in $D_5(G)$ are satisfied. Components of size two in $D_5(G)$ whose vertices are not in the same cluster and triangles in $D_5(G)$ will be satisfied unless they contain dangling vertices.

3. In STAGES 4, 5 and 6 (Section 7.4), we perform *global discharging* by transferring the aggregate charge from the set of reserved vertices to a set of unsatisfied degree-5 vertices. This provides sufficient charge to satisfy the singletons of $D_5(G)$ and the remaining dangling vertices.

7.1 Triggered Discharging

In STAGE 1, vertices of degree at least 8 discharge to their degree-5 neighbors as follows.

Rule 1 Every vertex v with $d(v) \geq 8$ and r neighbors of degree 5 sends $\frac{ch_0(v)}{r}$ to each neighbor of degree 5.

In STAGE 2, vertices of degree $6, 7$ discharge when specific conditions are met. STAGE 2 terminates when no vertex triggers a condition to discharge.

Rule 2A Every vertex v with $d(v) = 7$ and $r \leq 5$ unhappy upward neighbors sends $\frac{ch_0(v)}{r}$ to each unhappy upward neighbor.

Rule 2B Every vertex v with $d(v) = 6$ neighboring exactly one unhappy upward cluster sends $2 - \varepsilon$ to that cluster.

Define the function

$$\psi(d, r) = \frac{5d - 28 - \varepsilon}{r}$$

to be the amount of initial charge sent by a vertex of degree d to r neighbors of degree 5. A simple calculation yields the following lemma.

Lemma 7.3. *All of the following hold.*

1. *For fixed $d \geq 1$, $\psi(d, r)$ is decreasing in r.*
2. *For fixed $r \geq 1$, $\psi(d, r)$ is increasing in d.*
3. *For a fixed $0 \leq i \leq 5$, $\psi(d, d - i)$ is increasing in d.*
4. *A vertex v of degree $d \geq 8$ sends at least $\frac{3}{2} - \frac{1}{8}\varepsilon$ charge in* STAGE 1 *to each neighbor of degree 5.*
5. *A vertex v of degree $d \geq 8$ and $r < d$ neighbors of degree 5 always sends at least $\psi(8, 7) = \frac{12}{7} - \frac{1}{7}\varepsilon$ charge to each neighbor of degree 5.*
6. *A vertex v of degree 7 sends at least $\psi(7, 5) = \frac{7}{5} - \frac{1}{5}\varepsilon$ charge in* STAGE 2 *to each neighbor of degree 5 to which v sends charge.*

In particular, note $\psi(8, 8) \geq \psi(7, 5)$, so the minimum amount of charge sent to a vertex by another vertex in STAGES 1, 2 *is $\psi(7, 5) = \frac{7}{5} - \frac{1}{5}\varepsilon$.*

Proof. This follows from elementary calculations; see Appendix A.5. □

Remark. *To simplify the presentation, we adopt the following convention. When we make a statement such as "v triggers* RULE 2A *and sends charge to C", it should be understood that the discharge only occurs if the receiving cluster is not already happy. This way, we avoid repeatedly stating the condition 'if C is still unhappy' before every invocation of a discharging rule.*

Lemma 7.4. *Let H be a proto-gadget of G, and $C \subseteq V(H)$ a cluster. Then C is happy after* STAGE 2.

Proof. This proof is conceptually straightforward, but the detailed calculations are lengthy, so we provide a sketch here and defer the full proof to Appendix A.5.

Using Lemma 5.11, we may reduce to the case where C is part of the frame $V^*(H)$, and we will show that C receives sufficient charge from its neighbors in $V^*(H)$. Every vertex of $V^*(H)$ is linked to every other vertex by real or replacement edges. If $v \in V^*(H)$ has $d_G(v) \geq 8$, then v sends a large amount of charge to C. For v with $d_G(v) \leq 7$, we argue that sufficiently many such v have few neighbors of degree 5 and therefore trigger RULE 2. The key is that v does not send charge into replacement edges by Corollary 6.16, and by counting degrees, we can show v has few neighbors of degree 5 outside H, so RULE 2 is triggered. Thus, we can show C receives sufficient charge from $V^*(H)$. □

Corollary 7.5. *If x is a vertex of degree 5 and x is unhappy after* STAGE 2*, then x is not in any proto-gadget and hence x is upward of all its neighbors.*

Proof. By Lemma 7.4, all clusters contained in proto-gadgets are happy after STAGE 2. Thus, x is not in a proto-gadget. So by the definition of downward and upward neighbors, x is upward of its neighbors. □

Corollary 7.6. *If w is a vertex of degree 6 such that w is a split vertex of a split k-Ore subgraph H of G, then the downward neighbors of w in H are happy after* STAGE 2.

Proof. Let x be a degree-5 neighbor of w such that $x \in V(H)$. As G is not k-Ore, we find that x is not a split vertex of H. By Lemma 5.11, there exists a proto-gadget H' contained in H with $x \in V^*(H')$. By Lemma 7.4, x is happy after STAGE 2 as desired. □

Corollary 7.7. *If (x, y, z) is an induced path in $D_5(G)$, then x and z are happy after* STAGE 2*, that is, $ch_2(x), ch_2(z) \geq 2 + 2\varepsilon$.*

Proof. By Lemma 5.4, $G_{y\to x}$ and $G_{y\to z}$ contain 6-Ore subgraphs, since z, x have degree 4 after deleting y. Hence there exist gadgets H_1, H_2 of G with $x \in V^*(H_1), z \in V^*(H_2)$. Since x, z are not in clusters of size 2, Lemma 7.4 implies that $ch_2(x), ch_2(z) \geq 2 + 2\varepsilon$ as desired. □

7.2 Discharging: Second Stage

We next carefully discharge the remaining degree 6 and 7 vertices.

Rule 3A : Every vertex v with $d(v) \geq 7$ distributes all remaining positive charge equally to its degree-5 neighbors.

Rule 3B : Every vertex v with $d(v) = 6$ adjacent to at most 2 unhappy neighbors distributes its charge equally to those neighbors. If the first condition is not met, and w is adjacent to exactly three upward neighbors x, y, z of degree 5 with $xy \in E(G)$ and $xz, yz \notin E(G)$, then w sends $1 - \frac{1}{2}\varepsilon$ charge to z and $\frac{1}{2} - \frac{1}{4}\varepsilon$ charge to each of x, y[5]

A degree-6 vertex which has not discharged by the end of STAGE 3 will be called a *reserved vertex*. We define the *reserve degree* of a vertex $x \in D_5(G)$ to be the number of neighbors of x which are reserved and denote this by $r(x)$.

Vertices of $D_5(G)$ which do not neighbor too many reserved vertices will be satisfied after STAGE 3. We will handle singletons separately using global discharging at the end. The remaining cases, components of $D_5(G)$ of size greater than 1 with multiple reserved neighbors, represent the most difficult case to discharge, owing to a paucity of nearby charge as well as relatively weak local structure. We study that local structure below.

Definition 7.8. *A* dangling vertex *is a vertex $x \in D_5(G)$ which satisfies the following properties:*

- *x has a neighbor $y \in D_5(G)$, but x, y are not in the same cluster of G.*
- *$G_{x \to y}$ is 2-tight and ungemmed.*
- *There exists a kite H with $V^*(H) \cap D_5(G) = \{x\}$ and $y \notin V(H)$. If $w \in V^*(H)$ has $d_G(w) = 6$, then w is not adjacent to y.*

If x satisfies these conditions with respect to y, we say that x dangles from *y.*

Lemma 7.9. *Suppose that x dangles from y. Then all of the following hold:*

- *$r(x) \leq 2$, and*
- *if $|N(x) \cap D_5(G)| \geq 2$, then $r(x) \leq 1$, and*
- *if $r(x) \leq 1$ and $N(x) \cap D_5(G) = \{y\}$, then x is satisfied after STAGE 3 and $ch_3(x) \geq \frac{2}{5} - \frac{11}{5}\varepsilon$, and*
- *if either $r(x) = 2$, or $r(x) = 1$ and $|N(x) \cap D_5(G)| \geq 2$, then $ch_3(x) \geq -3 - \varepsilon + 2\psi(7, 5) = -\frac{1}{5} - \frac{7}{5}\varepsilon$.*

Proof. By the definition of dangles, we have that $x \in D_5(G)$, $xy \in E(G)$, $G_{x \to y}$ is 2-tight and ungemmed, and there exists a kite H with $V^*(H) \cap D_5(G) = \{x\}$. Furthermore by definition, if $w \in V^*(H)$ has $d_G(w) = 6$, then w is not adjacent to y. Let $V^*(H) = \{x, w_1, w_2, w_3\}$.

[5] If $xz \in E(G)$ or $yz \in E(G)$, then $\{x, y, z\}$ is a connected component of $D_5(G)$, which is handled by other means.

Note that since $d(x) = 5$ and G does not admit 2-edge additions, we have that x is not incident with a replacement edge of H. This follows from Lemma 6.3, since if x were incident with a replacement edge F of H, then $d_F(x) \geq 3$, in which case $d_G(x) \geq 6$ (since $w_1, w_2, y \in N(x) \setminus V(F)$), a contradiction. Hence for each $i \in \{1, 2, 3\}$, x is adjacent to w_i.

We first prove a series of claims which concerns the case where two of w_1, w_2, w_3 have degree 6 and are adjacent.

Claim 7.10. *Suppose that $d(w_1) = d(w_2) = 6$, and $w_1w_2 \in E(G)$. If $z \in (D_5(G) - H) \cap (N(w_1) \cup N(w_2))$ and $zy \in E(G)$, then $zx \notin E(G)$) and z is happy after* STAGE 2.

Proof. Suppose not. We may assume without loss of generality that $z \in N(w_1)$. First suppose that $zx \in E(G)$. Recall that by assumption on H and the definition of dangling, y is not adjacent to w_1. Hence if we consider $D_5(G_{z\to y})$, we find that $x, y, \tilde{y}, w_1 \in D_5(G_{z\to y})$. Thus $D_5(G_{z\to y})$ has a component of size at least 4. So by Corollary 6.12, $G_{z\to y}$ is not both 2-tight and ungemmed.

By Lemma 5.3, $G_{z\to y}$ is not 5-colorable and hence contains a 6-critical subgraph H'. Since G is tight and ungemmed, we have by Lemma 5.7 that $G_{z\to y}$ is ungemmed. If $H' = G_{z\to y}$, it follows that $G_{z\to y}$ is 2-tight and ungemmed, a contradiction to what was proven above. Thus, we may assume $H' \neq G_{z\to y}$.

By Lemma 5.4, H' is 6-Ore. By Lemma 5.9, we have that $y, \tilde{y} \in V^*(H')$. By counting degrees, we deduce that $x, w_1, w_2 \in V^*(H')$ as well, whence $yw_1 \in E(G)$ (again counting degrees with Lemma 6.3), a contradiction.

So we may assume that $zx \notin E(G)$. Hence xyz is an induced path in $D_5(G)$, and by Corollary 7.7, z is happy after STAGE 2, a contradiction. □

Claim 7.11. *Suppose that $d(w_1) = d(w_2) = 6$, and $w_1w_2 \in E(G)$. If $z \in (D_5(G) - H) \cap (N(w_1) \cup N(w_2))$ and $zx, zy \notin E(G)$, then every vertex in $D_5(G) \cap (N(w_1) \cup N(w_2)) \setminus \{x, z\}$ is happy after* STAGE 2.

Proof. Suppose not. That is, there exists a vertex $z' \in D_5(G) \cap (N(w_1) \cup N(w_2)) \setminus \{x, z\}$ that is unhappy after STAGE 2. Hence the conclusion of Claim 7.10 does not apply to z'. This implies that $yz' \notin E(G)$. Hence $z' \in D_5(G_{x\to y})$. Similarly since $zy \notin E(G)$, we have that $z \in D_5(G_{x\to y})$. Thus $\{w_1, w_2, z, z'\}$ are in a component of size at least 4 in $G_{x\to y}$.

Since x dangles from y by hypothesis, we have by definition of dangling that $G_{x\to y}$ is 2-tight and ungemmed, contradicting Corollary 6.12 as there is a component of $D_5(G_{x\to y})$ of size at least 4. □

Claim 7.12. *If $d(w_1) = d(w_2) = 6$ and $w_1w_2 \in E(G)$, then at most two vertices in $(D_5(G) - H) \cap (N(w_1) \cup N(w_2))$ are unhappy after* STAGE 2*, and if there are two, then at least one of them is adjacent to x.*

Proof. Suppose not. Let Z be the set of vertices in $(D_5(G) - H) \cap (N(w_1) \cup N(W_2))$ that are unhappy after STAGE 2. Hence either $|Z| \geq 3$, or $|Z| = 2$ and $|Z \cap N(x)| = 0$. For each $z \in Z$, it follows Claim 7.10 that $zy \notin E(G)$

since z is unhappy after STAGE 2. Then by Claim 7.11 since $|Z| \geq 2$, it follows that $zx \in E(G)$ for each $z \in Z$. Thus $Z \subseteq N(x)$. Since $|N(x)| = 5$ and $w_1, w_2, y \in N(x) \setminus Z$, we find that $|Z| \leq 2$. Hence $|Z| = 2$ and yet $|Z \cap N(x)| = 2$, a contradiction. □

We now proceed with the main argument. We first argue that $r(x) \leq 2$. If x neighbored three reserved vertices, then at least two of w_1, w_2, w_3 are reserved; we may assume without loss of generality that w_1 and w_2 are reserved.

Claim 7.13. *If both w_1, w_2 are reserved, then w_1, w_2 are adjacent.*

Proof. Suppose not. Hence there is a replacement edge $e = w_1w_2$ of H. In the latter case, Lemma 6.3 implies that w_1 has at least three neighbors inside e, and thus at most one upward neighbor of degree 5 not in H. By Lemma 5.11, the degree-5 neighbors of w_1 inside e are contained in proto-gadgets. Therefore by Lemma 7.4 and Corollary 7.6, the degree 5-neighbors of w inside e are happy after STAGE 2. Hence w_1 is adjacent to at most two neighbors that are unhappy after STAGE 2, which contradicts w_1 being reserved. Thus, w_1 is not incident to a replacement edge, a contradiction. □

Claim 7.14. *At least one of w_1 or w_2 is not reserved.*

Proof. Suppose not. That is w_1 and w_2 are reserved. By Claim 7.13, w_1 and w_2 are adjacent. Since w_1 is reserved, by definition we have that $|N(w_1) \cap D_5(G) \setminus \{x\}| \geq 2$ and at least two vertices in $N(w_1) \cap D_5(G) \setminus \{x\}$ are unhappy after STAGE 2, call them z_1 and z_2. By Claim 7.10, we find that $z_1, z_2 \notin N(y)$. It then follows that $z_1, z_2 \in D_5(G_{x \to y})$. But then $\{z_1, z_2, w_1, w_2\}$ are in a component of size at least 4 in $G_{x \to y}$, contradicting Corollary 6.12 because $G_{x \to y}$ is 2-tight and ungemmed by hypothesis. □

This claim completes the proof that $r(x) \leq 2$. Moreover, if $|N(x) \cap D_5(G)| \geq 2$, then the neighbor of x outside H has degree 5 and is therefore not a reserved vertex. At most one of the vertices $w_1, w_2, w_3 \in V^*(H)$ can be reserved, by Claim 7.14, so $r(x) \leq 1$.

Claim 7.15. *If at least two of w_1, w_2, w_3 (wlog, w_1, w_2) satisfy $d(w_1) = d(w_2) = 6$, and $N(x) \cap D_5(G) = \{y\}$, then $r(x) \leq 1$ and $ch_3(x) \geq \frac{2}{5} - \frac{11}{5}\varepsilon$.*

Proof. The proof is divided into cases depending on $d(w_3)$.

Case A If $d(w_3) = 6$, then by Lemma 6.3, there can be at most one replacement edge in H. Hence, we deduce that $\{w_1, w_2, w_3\}$ induces a clique in G, and is therefore in a connected component of $D_5(G_{x \to y})$. By Corollary 6.12, we have that $w_1w_2w_3$ is a component of $D_5(G_{x \to y})$. Let $S = D_5(G) \cap ((N(w_1) \cup N(w_2) \cup N(w_3)) \setminus \{w_1, w_2, w_3, x\})$. It follows that S is a subset of $N(y)$, and since $|N(y) \cap D_5(G) \setminus \{x\}| \leq 1$, we find that $|S| \leq 1$.

Suppose $|S| = 1$ and let $z \in S$. If $yz \notin E(G)$, then $\{w_1, w_2, w_3, z\}$ is a component of size 4 in $D_5(G_{x \to y})$, a contradiction. Hence, we must have $yz \in E(G)$. Our other assumption on $N(x) \cap D_5(G)$ implies that $xz \notin E(G)$, so (x, y, z) is an induced path in $D_5(G)$. By Corollary 7.7, x is happy after STAGE 2.

So we may assume that $S = \emptyset$. Then each of w_1, w_2, w_3 is adjacent to at most one unhappy vertex (namely, x itself), and x receives $2 - \varepsilon$ charge from each until x is happy.

Case B Suppose $d(w_3) \geq 7$.

First, consider the case where a replacement edge e joins w_1, w_2. Lemma 6.3 then implies that $w_1w_3, w_2w_3 \in E(G)$, so w_3 has at most 5 unhappy upward neighbors and therefore triggers in RULE 2A. Lemma 7.3 implies that x receives at least $\psi(7, 5)$ from w_3. For w_1 and w_2, Lemma 5.11 and Corollary 7.6 imply that degree-5 neighbors inside e are happy after STAGE 2. Counting degrees, w_1 and w_2 can have at most one degree-5 neighbor outside H, and thus at most two unhappy neighbors (one being x). It follows that the conditions for RULE 3B apply and w_1, w_2 each send at least $1 - \frac{1}{2}\varepsilon$ charge to x. Thus, $ch_3(x) \geq 2(1 - \frac{1}{2}\varepsilon) + \psi(7, 5) - 3 - \varepsilon = \frac{2}{5} - \frac{11}{5}\varepsilon$. If $w_1w_2 \in E(G)$, then by Claim 7.12, there is at most one unhappy vertex in $(D_5(G) - H) \cap (N(w_1) \cup N(w_2))$ since by assumption $N(x) \cap D_5(G) \setminus \{y\} = \emptyset$. Thus, x receives at least $2 - \epsilon$ charge from w_1 and w_2 together, and we again have $ch_3(x) \geq \frac{2}{5} - \frac{11}{5}\varepsilon$.

□

To complete the proof, it remains to consider the case where at least two vertices of $V^*(H)$ have degree at least 7.

Claim 7.16. *Suppose that $d(w_1), d(w_2) \geq 7$. Let α_3 denote the charge sent to x by w_3. Then x receives at least $2\psi(7, 5) + \alpha_3$ charge from $V^*(H)$.*

Proof. It suffices to show that w_1, w_2 trigger by STAGE 2. Since $w_1 \in V^*(H)$, w_1 can have at most $d(w_1) - 2$ upward neighbors, and therefore meets the conditions of RULE 1 or RULE 2. It follows that w_1 sends at least $\psi(7, 5)$ charge to x, and likewise for w_2. □

Let ν denote the charge sent to x by its neighbor $\hat{z}$ outside H.

Claim 7.17. *Suppose that $d(w_1), d(w_2) \geq 7$. Then we either have $2\psi(7, 5) + \alpha_3 + \nu \geq \frac{2}{5} - \frac{11}{5}\varepsilon$, or one of the following two conditions holds:*

1. *$r(x) = 2$, or*
2. *$r(x) = 1$ and $|N(x) \cap D_5(G)| \geq 2$.*

Proof. If $r(x) \leq 1$, then x receives charge from at least one of w_3 or $\hat{z}$. By the definition of RULE 3, we find that $\alpha_3 + \nu \geq 1 - \frac{1}{2}\varepsilon$ and hence $ch_3(x) \geq 2\psi(7, 5) + 1 - \frac{1}{2}\varepsilon - 3 - \varepsilon = \frac{2}{5} - \frac{11}{5}\varepsilon$.

Suppose $\alpha_3 + \nu = 0$. Then $ch_3(x) \geq 2\psi(7, 5) - 3 - \varepsilon = -\frac{1}{5} - \frac{7}{5}\varepsilon$. This can only occur if *neither* w_3 nor $\hat{z}$ sends charge to x, which implies that either:

1. Both w_3 and $\hat{z}$ are reserved, or
2. w_3 is reserved and $d_G(\hat{z}) = 5$.

□

Combining Claim 7.15 and Claim 7.16, the proof of Lemma 7.9 is complete.

□

The argument for the next two lemmas is similar to that of Lemma 7.9 but substantially simpler, so we defer it to Appendix A.5.

Lemma 7.18. *Let $S = \{x, y\}$ be a component of size 2 in $D_5(G)$ such that S is not a cluster. If both $G_{y\to x}$ and $G_{x\to y}$ are 2-tight and ungemmed, and $|N(x) \cap N(y) \cap D_6(G)| = 0$, then at least one of the following holds:*

1. *x is happy after* STAGE *3, or*
2. *Every neighbor of x (other than y) sends at least $1 - \frac{1}{2}\varepsilon$ charge to x, or*
3. *y dangles from x.*

The same applies to y.

Lemma 7.19. *Let $S = \{x, y\}$ be a component of size 2 in $D_5(G)$ such that S is not a cluster. If both $G_{y\to x}$ and $G_{x\to y}$ are 2-tight and ungemmed, and $|N(x) \cap N(y) \cap D_6(G)| = 1$, then y dangles from x, and x dangles from y.*

Finally, we use Lemma 7.9 to analyze triangles in $D_5(G)$.

Lemma 7.20. *Let $S = \{x_1, x_2, x_3\}$ be a triangle in $D_5(G)$, not containing a cluster of size 2. Then $G_{x_j \to x_i}$ is 2-tight and ungemmed for any two vertices of $x_i, x_j \in S$, and all of the following hold:*

1. *no two vertices of S have a common neighbor having degree 6, and*
2. *each vertex x of S has $r(x) \leq 1$, and*
3. *if w is a reserved vertex neighboring a vertex of S, then the other degree-5 neighbors of w are singletons in $D_5(G)$, and*
4. *if ρ denotes the number of reserved vertices adjacent to at least one vertex of S, then*

$$ch_3(S) \geq \left(-\frac{1}{5} - \frac{7}{5}\varepsilon\right)\rho + \left(\frac{1}{5} - \frac{11}{5}\varepsilon\right)(3 - \rho).$$

Proof. We first prove the first statement that for every $i \neq j \in \{1, 2, 3\}$, $G_{x_j \to x_i}$ is 2-tight and ungemmed. Suppose not. Without loss of generality, suppose $G_{x_2 \to x_1}$ is not 2-tight or not ungemmed. Since x_3 is adjacent to both x_1, x_2, the vertex x_3 has degree 5 in $G_{x_2 \to x_1}$. From Lemma 5.4, it Follows that $G_{x_2 \to x_1}$ contains a 6-Ore subgraph H. By counting degrees, we see that $x_1, \tilde{x_1}, x_3 \in V^*(H)$. Applying Corollary 6.5 to x_3, we find that x_3 is in the same cluster as

x_1 in $G_{x_2 \to x_1}$, and therefore in G, which contradicts the definition of S. This proves the first statement.

Next we prove that all of (1)-(4) hold. First suppose (1) does not hold, that is, x_1, x_2 have a common neighbor w of degree 6. $G_{x_2 \to x_1}$ then contains a cluster $\{x_1, \widetilde{x_1}\}$ with a neighbor x_3 having degree 5 and a neighbor w having degree 6, which contradicts Lemma 6.6. So we may assume that (1) holds.

Let $\{w_1, w_2, w_3\} = N(x_1) \setminus S$. Let $G' = G_{x_2 \to x_1}$ and consider $G'' = G'_{x_3 \to x_1}$. Note that G'' contains a cluster of size 3, so it contains a 6-Ore subgraph H with $\{x_1, w_1, w_2, w_3\} \subseteq V^*(H)$. Let H_1 be the kite contained in H with $V^*(H_1) = \{x, w_1, w_2, w_3\}$. Since x_1 has no common neighbors of degree 6 with x_2, x_3, and $x_2, x_3 \notin V(H_1)$, we see that x_1 dangles from both x_2, x_3, and therefore Lemma 7.9 applies to x_1. Symmetrically, there exist kites H_2 and H_3, so x_2 dangles from x_1, x_3 and x_3 dangles from x_1, x_2, so Lemma 7.9 applies to x_2 and x_3 as well.

First, notice that Lemma 7.9 implies x_1 has reserve degree at most one and hence (2) holds. Lemma 7.9 also implies that $ch_3(x_1) \geq \frac{1}{5} - \frac{11}{5}\varepsilon$ if x_1 neighbors no reserved vertices, and $ch_3(x_1) \geq -\frac{1}{5} - \frac{7}{5}\varepsilon$ if x_1 neighbors one reserved vertex. Using the same reasoning for x_2 and x_3, we sum the charges on S to obtain

$$ch_3(S) = ch_3(x_1) + ch_3(x_2) + ch_3(x_3) \geq \left(-\frac{1}{5} - \frac{7}{5}\varepsilon\right)\rho + \left(\frac{1}{5} - \frac{11}{5}\varepsilon\right)(3 - \rho)$$

and hence (4) holds.

Finally, to show that (3) holds, suppose that w is a reserved neighbor of x_1, and let $z_1 \in N(w) \cap D_5(G)$, $z_1 \neq x_1$. Letting $G' = G_{x_1 \to x_2}$, we find that w has degree 5 in $D_5(G')$, so the component T containing w in $D_5(G')$ has size at most 3 by Corollary 6.12. If z_1 is not a singleton of $D_5(G)$, then z_1 has a neighbor $z_2 \in D_5(G)$, so $T = \{w, z_1, z_2\}$, in which case $N(w) \cap D_5(G) \subseteq \{x_1, z_1, z_2\}$, with $z_1 z_2 \in E(G), x_1 z_1, x_1 z_2 \notin E(G)$. Then w satisfies the conditions of RULE 3B and is not reserved, a contradiction. □

Corollary 7.21. *If w is a reserved vertex that is a neighbor of a vertex in a triangle S of $D_5(G)$, then $|(N(w) \cap D_5(G)) \setminus S| \geq 2$ and the vertices $v \in (N(w) \cap D_5(G)) \setminus S$ are singletons in $D_5(G)$.*

7.3 Counting Charge

Using the results from the previous section, we can show that almost all components of $D_5(G)$ are satisfied after STAGE 3. Before completing the final stages, we perform an accounting of which vertices still require charge. The next three propositions follow immediately from Lemma 7.4 and its corollaries.

Proposition 7.22. *If C is a cluster of size 2 and C has no neighbor having degree 5, then $ch_3(C) \geq 0$.*

Proposition 7.23. *Let $C = \{x, y\}$ be a cluster with a neighbor z having degree 5, and let $S = \{x, y, z\}$. Then $ch_3(S) \geq -1 + \varepsilon$, and furthermore, $ch_3(S) \geq 0$ unless z has reserve degree at least* 3.

Proposition 7.24. *Let $S = (x, y, z)$ be an induced path in $D_5(G)$. Then $ch_3(S) \geq 0$.*

Proposition 7.25. *Let $S = \{x, y\}$ be a component of size 2 in $D_5(G)$, not in a cluster, and let $S' = \{v \in S : ch_3(v) < 0\}$. Let $\rho = \max_{v \in S'} r(v)$. Then $ch_3(S) \geq \min\{0, -\rho + \frac{3}{2}\}$. Furthermore, if $\rho \geq 3$, then we have a stronger bound that $ch_3(S) \geq -\rho + \frac{8}{5}$.*

Proof. Suppose that x is happy after STAGE 3, so $ch_3(x) \geq 2 + 2\varepsilon$. Since x and y have degree 5 and are adjacent, $\rho \leq 4$. We calculate that

$$ch_3(S) = ch_3(x) + ch_3(y) \geq 2 + 2\varepsilon - 3 - \varepsilon + (4 - \rho)\left(1 - \frac{1}{2}\varepsilon\right)$$
$$= -\rho + 3 - \left(1 - \frac{1}{2}\rho\right)\varepsilon \geq -\rho + \frac{8}{5}.$$

If $G_{y \to x}$ contains a 6-Ore subgraph, then x is in a proto-gadget and Lemma 7.4 implies that x is happy after STAGE 2. Therefore, by Lemmas 5.4, 5.7 and 5.8, we may assume that $G_{y \to x}$ and $G_{x \to y}$ are both 2-tight and ungemmed.

$G_{y \to x}$ contains a cluster $C = \{x, \widetilde{x}\}$ of size 2. Lemma 6.6 implies that C has at most 1 neighbor of degree 6. Such a neighbor of C is either a vertex of degree 5 in G adjacent to x but not y, or a vertex having degree 6 in G adjacent to both x and y. Since x does not have a neighbor having degree 5 that is distinct from y, we deduce that x, y have at most 1 common neighbor having degree 6. Therefore either Lemma 7.18 or Lemma 7.19 applies.

Suppose first that x, y have no common neighbors of degree 6. Apply Lemma 7.18 to the vertex x. If Lemma 7.18(1) holds, then x is happy after STAGE 3 and the proposition follows as above.

So suppose Lemma 7.18(2) holds. If x receives at least $1 - \frac{1}{2}\varepsilon$ charge from each neighbor, then x neighbors no reserved vertices and ρ is the number of reserved vertices neighboring y. The amount of charge received by y is at least $1 - \frac{1}{2}\varepsilon$ for each non-reserved vertex, so

$$ch_3(x) + ch_3(y) \geq -6 - 2\varepsilon + 4\left(1 - \frac{1}{2}\varepsilon\right) + (4 - \rho)\left(1 - \frac{1}{2}\varepsilon\right) = -\rho + 2 - \left(6 - \frac{1}{2}\rho\right)\varepsilon,$$

which is clearly greater than $-\rho + \frac{8}{5}$ as desired.

Hence we may assume that Lemma 7.18(3) holds, that is y dangles from x. Since Lemma 7.18 applies symmetrically to y, we find by symmemtry that x dangles from y. Now Lemma 7.9 implies that

1. x neighbors at most two reserved vertices.
2. $ch_3(x) \geq -\frac{1}{5} - \frac{7}{5}\varepsilon$.
3. If x neighbors at most one reserved vertex, then $ch_3(x) \geq \frac{1}{5} - \frac{11}{5}\varepsilon$.

By symmetry, the same applies to y, so $\rho \leq 2$. Thus, if neither x nor y neighbors two reserved vertices, then $ch_3(S) \geq 2(\frac{1}{5} - \frac{11}{5}\varepsilon) \geq 0$. If $\rho = 2$, then we have $ch_3(S) \geq 2(-\frac{1}{5} - \frac{7}{5}\varepsilon) \geq -\frac{1}{2} = -2 + \frac{3}{2}$, as desired.

If x, y have exactly one common neighbor having degree 6, then Lemma 7.19 implies that x dangles from y and vice versa. We have just shown that we have the desired result in this case.

Notice that in all the cases where $\rho \geq 3$ is possible, the stronger bound $ch_3(S) \geq -\rho + \frac{8}{5}$ holds. □

7.4 Global Discharging

We can now employ a global discharging argument to resolve the singletons and all other unsatisfied vertices.

Let U be the set of components S of $D_5(G)$ such that $ch_3(S) < 0$. From each component S of U, let v_S denote a vertex of S whose reserve degree is maximum over vertices in S. Let A denote this set of vertices; note that A is an independent set. Let B denote the set of reserved vertices neighboring A. We apply Corollary 3.3 to A and B, and deduce that $|E_G(A, B)| \leq |A| + 2|B|$.

We now perform the following rules in stages (namely we have separate STAGES 4, 5 AND 6).

Rule 4 : Add $-\frac{4}{5} - \frac{1}{5}\varepsilon$ charge to every vertex in A, and distribute $\left(\frac{4}{5} + \frac{1}{5}\varepsilon\right)|A|$ charge among the vertices of B. The total charge is unchanged.

Rule 5 : Send $\frac{4}{5} + \frac{1}{5}\varepsilon$ charge along every edge in $E(A, B)$ towards A.

Rule 6 : Redistribute the charge in B so that every vertex of B has at least $\frac{2}{5} - \frac{7}{5}\varepsilon$ charge. Every vertex in B adjacent to a triangle sends $\frac{2}{5} - \frac{7}{5}\varepsilon$ charge to that triangle.

First, we verify that B has sufficient charge stored after STAGE 5 to implement RULE 6. Observe that $ch_4(B) = (\frac{4}{5} + \frac{1}{5}\varepsilon)|A| + (2-\varepsilon)|B|$. By Corollary 3.3, the total charge sent in STAGE 5 from B to A is at most

$$\left(\frac{4}{5} + \frac{1}{5}\varepsilon\right)|E(A, B)| \leq \left(\frac{4}{5} + \frac{1}{5}\varepsilon\right)(|A| + 2|B|) = \left(\frac{4}{5} + \frac{1}{5}\varepsilon\right)|A| + \left(\frac{8}{5} + \frac{2}{5}\varepsilon\right)|B|$$

Consequently,

$$ch_5(B) \geq \left(\frac{4}{5} + \frac{1}{5}\varepsilon\right)|A| + (2 - \varepsilon)|B| - \left(\frac{4}{5} + \frac{1}{5}\varepsilon\right)|A| - \left(\frac{8}{5} + \frac{2}{5}\varepsilon\right)|B| \geq \left(\frac{2}{5} - \frac{7}{5}\varepsilon\right)|B|.$$

By Lemma 7.20, a vertex in B can be adjacent to at most one triangle. Thus, we have enough charge in B after STAGE 5 to carry out RULE 6. which ensures that $ch_6(B) \geq 0$.

Corollary 7.26. *A triangle neighboring ρ reserved vertices receives $(\frac{2}{5} - \frac{7}{5}\varepsilon)\rho$ charge in* STAGE 6*.*

Proof. Let w be a reserved vertex neighboring a triangle S. By Corollary 7.21, $w \in B$, and thus w sends charge to S in STAGE 6. □

We check that $ch_6(x) \geq 0$ for all vertices of degree 5.

Proposition 7.27. *All vertices are satisfied after* STAGE 6*.*

Proof. For vertices of degree at least 7, this follows immediately as our discharging rules never send more than the initial charge. Vertices of degree 6 also do not send more than their initial charge unless they belong to B. We have shown that $ch_6(B) \geq 0$. It only remains to check the charge of components of $D_5(G)$. From Corollary 6.12, there are six possible components of $D_5(G)$. We verify that each type is satisfied.

Singletons: A singleton $x \in D_5(G)$ has $ch_4(x) = -3 - \frac{4}{5} - (1 + \frac{1}{5})\varepsilon$. It receives at least $1 - \frac{1}{2}\varepsilon$ charge from its neighbors of degree at least 7, and $\frac{4}{5} + \frac{1}{5}\varepsilon$ from its neighboring reserved vertices. Therefore, it receives at least $\frac{4}{5} - \frac{1}{5}\varepsilon$ charge from every neighbor and since $\varepsilon \leq 1$,

$$ch_5(x) \geq -3 - \frac{4}{5} - \left(1 + \frac{1}{5}\right)\varepsilon + 5\left(\frac{4}{5} + \frac{1}{5}\varepsilon\right) = \frac{1}{5} - \frac{1}{5}\varepsilon \geq 0.$$

Induced Paths of Length 2: Let $S = (x, y, z)$ be an induced path of length 2 in $D_5(G)$. By Proposition 7.24, $ch_6(S) \geq ch_3(S) \geq 0$.

Clusters of Size 2: Let $C = \{x, y\}$ be a cluster of size 2. By Proposition 7.22, $ch_6(C) \geq ch_3(C) \geq 0$.

Clusters of Size 2 with a Degree 5 Neighbor: Let $S = \{x, y, z\}$, with $C = \{x, y\}$ a cluster of size 2. By Proposition 7.23, $ch_3(S) \geq 0$ unless z neighbors 3 reserved vertices, in which case $ch_3(S) \geq -1 + \varepsilon$. Then $z \in A$ and the reserved vertices neighboring z are in B, so z receives $3(\frac{4}{5} + \frac{1}{5}\varepsilon)$ charge in STAGE 5. Thus, since $\varepsilon \geq 0$,

$$ch_6(S) \geq ch_3(S) - \frac{4}{5} - \frac{1}{5}\varepsilon + 3\left(\frac{4}{5} + \frac{1}{5}\varepsilon\right) \geq \frac{3}{5} + \frac{8}{5}\varepsilon > 0.$$

Two Adjacent Vertices: Let $S = \{x, y\}$ be a component of $D_5(G)$, not in a cluster, with $ch_3(S) < 0$. We may assume that $ch_3(x) < 0$ and without loss of generality that the reserve degree of x is at least as large as the reserve degree of y. Then $x \in A$, and the ρ reserved vertices neighboring x are in B, so x receives at least $(\frac{4}{5} + \frac{1}{5}\varepsilon)\rho$ charge in STAGE 5. By Proposition 7.25, we find that $ch_3(S) \geq -\rho + \frac{3}{2}$, and if $\rho \geq 3$, then $ch_3(S) \geq -\rho + \frac{8}{5}$. If $\rho \leq 2$, then since $\varepsilon \leq \frac{3}{2}$, we have

$$\begin{aligned} ch_6(S) \geq \left(-\rho + \frac{3}{2}\right) - \frac{4}{5} - \frac{1}{5}\varepsilon + \left(\frac{4}{5} + \frac{1}{5}\varepsilon\right)\rho &= \frac{7}{10} - \frac{1}{5}\rho + \frac{\rho - 1}{5}\varepsilon \\ &\geq \frac{7}{10} - \frac{2}{5} + \frac{\rho - 1}{5}\varepsilon \geq 0. \end{aligned}$$

If $3 \leq \rho \leq 4$, we have a similar calculation:

$$ch_5(S) \geq \frac{4}{5} - \frac{1}{5}\rho + \frac{\rho - 1}{5}\varepsilon \geq \frac{\rho - 1}{5}\varepsilon \geq 0.$$

Triangle: Let S be a triangle, and let ρ be the number of reserved vertices that have a neighbor in S. By Lemma 7.20, $ch_3(S) \geq \left(-\frac{1}{5} - \frac{7}{5}\varepsilon\right)\rho + \left(\frac{1}{5} - \frac{11}{5}\varepsilon\right)(3-\rho)$. Thus, if $ch_3(S) < 0$, $\rho \geq 2$. By Corollary 7.21, a reserved vertex w neighboring S is also adjacent to a singleton of $D_5(G)$ that is unsatisfied after STAGE 3. Such a singleton is in A, so $w \in B$.

Let x be the vertex of S chosen to be in A. In STAGE 4, x receives $-\frac{4}{5} - \frac{1}{5}\varepsilon$ charge, and then receives $\frac{4}{5} + \frac{1}{5}\varepsilon$ charge back from its adjacent reserved vertex in STAGE 5. Hence $ch_5(S) = ch_3(S)$. Since every reserved vertex neighboring S is also in B, by the reasoning in the previous paragraph, S receives $(\frac{2}{5} - \frac{7}{5}\varepsilon)\rho$ charge from RULE 6. Hence

$$\begin{aligned} ch_6(S) \geq ch_5(S) + \left(\frac{2}{5} - \frac{7}{5}\varepsilon\right)\rho &= -\frac{2}{5}\rho + \frac{3}{5} - \frac{33}{5}\varepsilon + \frac{4\rho}{5}\varepsilon + \left(\frac{2}{5} - \frac{7}{5}\varepsilon\right)\rho \\ &= \frac{3}{5} - \frac{3\rho}{5}\varepsilon - \frac{33}{5}\varepsilon \geq 0, \end{aligned}$$

since $\rho \leq 3$ and $\varepsilon \leq \frac{1}{14}$.

□

Therefore, the total charge on G after STAGE 5 is non-negative. However, the total charge is invariant and the initial charge was $-p(G) - \delta T(G) < 0$, a contradiction. This completes the proof of Theorem 1.5.

References

[1] O. Borodin, Z. Dvořák, A. Kostochka, B. Lidický, and M. Yancey, *Planar 4-critical graphs with four triangles*, European Journal of Combinatorics, 41 (2014), pp. 138–151.

[2] G. Chen, P. Erdös, A. Gyárfás, and R. Schelp, *A class of edge critical 4-chromatic graphs*, Graphs and Combinatorics, 13 (1997), pp. 139–146.

[3] G. Dirac, *Theorem of R. L. Brooks and a conjecture of H. Hadwiger*, Proceedings of the London Mathematical Society, 3 (1957), pp. 161–195.

[4] T. Gallai, *Kritische graphen II*, Publications of the Mathematical Institute of the Hungarian Academy of Sciences, 8 (1963), pp. 373–395.

[5] R. Gould, V. Larsen, and L. Postle, *Structure in k-critical graphs*, manuscript.

[6] H. A. Kierstead and L. Rabern, *Extracting list colorings from large independent sets*, Journal of Graph Theory, 86 (2017), pp. 315–328.

[7] A. Kostochka and B. Reiniger, *The minimum number of edges in a 4-critical graph that is bipartite plus 3 edges*, European Journal of Combinatorics, 46 (2015), pp. 89–94.

[8] A. Kostochka and M. Stiebitz, *Colour-critical graphs with few edges*, Discrete Mathematics, 191 (1998), pp. 125–137.

[9] ——, *On the number of edges in colour-critical graphs and hypergraphs*, Combinatorica, 20 (2000), pp. 521–530.

[10] ——, *A new lower bound on the number of edges in colour-critical graphs*, Journal of Combinatorial Theory, Series B, 87 (2003), pp. 374–402.

[11] A. Kostochka and M. Yancey, *Ore's conjecture on color-critical graphs is almost true*, Journal of Combinatorial Theory, Series B, 109 (2014), pp. 73–101.

[12] ——, *A Brooks-type result for sparse critical graphs*, Combinatorica, 38 (2018), pp. 887—-934.

[13] M. Krivelevich, *On the minimal number of edges in color-critical graphs*, Combinatorica, 17 (1997), pp. 401–426.

[14] V. Larsen, *An Epsilon Improvement to the Asymptotic Density of k-Critical Graphs*, PhD thesis, Emory University, 2015.

[15] C.-H. Liu and L. Postle, *On the minimum edge-density of 4-critical graphs of girth five*, Journal of Graph Theory, 86 (2017), pp. 387–405.

[16] Ø. Ore, *The Four Color Problem*, Academic Press, New York, 1967.

[17] L. Postle, *The edge-density of 4-critical graphs of girth 5 is 5/3 +* ϵ, manuscript.

[18] ——, *On the minimum edge-density of 5-critical triangle-free graphs*, Electronic Notes in Discrete Mathematics, 49 (2015), pp. 667–673.

[19] ——, *On the minimum edge-density of 5-critical triangle-free graphs*, `https://arxiv.org/abs/1602.03098`, (2016). Extended version of [18].