A COMPLICATED SOLUTION TO AN EASY PROBLEM

WENBO GAO

Abstract. We give a convoluted solution to an easy linear algebra problem using needlessly complicated arguments from algebraic combinatorics.

Let $v \in \mathbb{R}^n$, and let $V = \mathrm{Diag}(v)$, the diagonal matrix with entries $v_1, \ldots, v_n$. Consider the matrix $\mathcal{V}$ given by

$$\mathcal{V} = nV^2 - vv^T$$

We aim to show that $\mathcal{V} \succeq 0$. The astute reader will find it an easy exercise in linear algebra. However, that would defeat the purpose of this article.

Consider a more general matrix $\mathcal{V}_t = tV^2 - vv^T$. Recall that a symmetric matrix is positive semidefinite if and only if the determinant of every symmetric minor is non-negative. Let $\mathcal{V}_t[I]$ denote the minor induced by the rows and columns in the index set I. With this notation, we must prove that

$$\det(\mathcal{V}_t[I]) \geq 0 \quad \forall I \subseteq \{1, \ldots, n\}$$

Let $|I| = m$; without loss of generality, we may assume that $I = \{1, \ldots, m\}$. The matrix $\mathcal{V}_t[I]$ has the form

$$\mathcal{V}_t[I] = \begin{pmatrix} (t-1)v_1^2 & -v_1v_2 & \ldots & -v_1v_m \\ -v_2v_1 & (t-1)v_2^2 & \ldots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -v_mv_1 & \ldots & \ldots & (t-1)v_m^2 \end{pmatrix}$$

Using the Leibniz expansion of the determinant, we therefore have

$$\det(\mathcal{V}_t[I]) = \sum_{\sigma \in \mathcal{S}_m} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^{m} \mathcal{V}_t[I](i, \sigma(i))$$

where $\mathcal{S}_m$ is the symmetric group on m elements.

The product $\prod_{i=1}^{n} \mathcal{V}_t[I](i, \sigma(i))$ has a simple form. Since σ is a permutation, $\sigma(i)$ ranges over $\{1, \ldots, m\}$ as i ranges over $\{1, \ldots, m\}$. The $(i, \sigma(i))$ entry of $\mathcal{V}_t[I]$ is simply $-v_i v_{\sigma(i)}$ if $i \neq \sigma(i)$, and is equal to $(t-1)v_i^2$ otherwise. Let $\mathrm{fix}(\sigma)$ denote the number of fixed points of $\sigma \in \mathcal{S}_m$. We therefore have

$$\det(\mathcal{V}_t[I]) = \sum_{\sigma \in \mathcal{S}_m} (-1)^{\mathrm{sgn}(\sigma) + m - \mathrm{fix}(\sigma)} (t-1)^{\mathrm{fix}(\sigma)} \prod_{i=1}^{m} v_i^2$$

Let $\gamma = \prod_{i=1}^{m} v_i^2$, and note $\gamma \geq 0$. This constant appears in every term of the sum, so what we really need to compute is

$$\sum_{\sigma \in \mathcal{S}_m} (-1)^{\mathrm{sgn}(\sigma) + m - \mathrm{fix}(\sigma)} (t-1)^{\mathrm{fix}(\sigma)}$$

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Moreover, we have the following relation between $\mathrm{sgn}(\sigma)$ and $\mathrm{fix}(\sigma)$. If $C_1, \dots, C_r$ are the non-trivial cycles (i.e, with length at least 2) of σ, then modulo 2,

$$\mathrm{sgn}(\sigma) \equiv \sum_{i=1}^{r} |C_i| + 1 \equiv m - \mathrm{fix}(\sigma) + r$$

so we can compute

$$\sum_{\sigma \in \mathcal{S}_m} (-1)^{\mathrm{cyc}(\sigma)} (t-1)^{\mathrm{fix}(\sigma)}$$

where $\mathrm{cyc}(\sigma)$ denotes the number of non-trivial cycles in σ. This is where we can apply the machinery of combinatorics.

Consider the *combinatorial species* $\mathcal{P}$ of permutations on a finite set, with weights $\mathrm{fix}(\sigma)$ and $\mathrm{cyc}(\sigma)$. The exponential generating function of $\mathcal{P}$ is given by

$$\mathcal{F}(x, y, z) = \sum_{k=1}^{\infty} \left(\sum_{\sigma \in \mathcal{S}_k} y^{\mathrm{fix}(\sigma)} z^{\mathrm{cyc}(\sigma)} \right) \frac{x^k}{k!}$$

We can explicitly write down $\mathcal{F}(x, y, z)$ in this case. By the standard decomposition, a permutation is a set of cycles. The exponential generating function for the species $\mathcal{C}$ of cyclic permutations is well-known; it's simply

$$\mathcal{G}(x) = \sum_{k=1}^{\infty} \frac{x^k}{k} = -\log(1-x)$$

To account for the weights, notice that each cyclic permutation of size one contributes 1 to $\mathrm{fix}(\sigma)$, and each cyclic permutation of length at least 2 contributes 1 to $\mathrm{cyc}(\sigma)$. Thus, we have

$$\mathcal{G}(x, y, z) = yx + z \sum_{k=2}^{\infty} \frac{x^k}{k} = -z \log(1-x) - zx + yx$$

and

$$\mathcal{F}(x, y, z) = \exp \mathcal{G}(x, y, z) = e^{(y-z)x} (1-x)^{-z}$$

In our problem, we have $y = t - 1$ and $z = -1$, and we are interested in the coefficient of x^m. We find that

$$\begin{aligned} m![x^m]\mathcal{F}(x, y, z) &= m![x^m](1-x)e^{tx} \\ &= m! \left(\frac{t^m}{m!} - \frac{t^{m-1}}{(m-1)!} \right) \\ &= t^{m-1}(t-m) \end{aligned}$$

Thus, we finally obtain

$$\det(\mathcal{V}_t[I]) = \gamma t^{m-1}(t-m)$$

Our original problem concerned the case $t = n$. We immediately find that

$$\det(\mathcal{V}_n[I]) = \gamma n^{m-1}(n-m) \geq 0$$

for all $m \leq n$. Thus, $\mathcal{V}_n \succeq 0$.

Other information about $\mathcal{V}_t$ can easily be deduced from this formula. For instance, we find that $\det(\mathcal{V}_n) = 0$, so $\mathcal{V}_n$ is singular. However, if v has no zero entries, then deleting the first row and column leaves a matrix with determinant $\gamma n^{n-2} > 0$, so $\mathrm{rank}(\mathcal{V}_n) = n - 1$.