EQUIVALENT SIMPLICIAL POLYTOPES WITH INTEGER VERTICES

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Abstract. We give a complete proof that every simplicial polytope is combinatorially equivalent to a polytope with integer vertices.

The following result is well-known in the folklore of polytope theory:

Theorem 1. *For every simplicial polytope, there is a combinatorially equivalent polytope with integer vertex coordinates.*

The intuition for this theorem is straightforward: one can slightly 'perturb' the coordinates of each vertex to select rational points, while preserving the combinatorial structure of the polytope (scaling then produces integer coordinates). It is then rather surprising that Theorem 1 does *not* hold for general polytopes; Perles constructed an 8-dimensional polytope with 12 vertices which cannot be realized by any rational geometric polytope.

Though intuitive, the detailed proof of Theorem 1 is somewhat technically involved. Ziegler [1], in his brief discussion of this result, only comments that it follows from perturbation. We now provide the missing details.

While one can presumably argue directly about simplicial polytopes, it appears to be easier to prove Theorem 1 for *simple polytopes* and then invoke duality.

Theorem 2. *For every simple polytope, there is a combinatorially equivalent* $\mathcal{H}$*-polytope* $\{x \in \mathbb{R}^d : Ax \leq b\}$ *with* $A \in \mathbb{Q}^{m \times d}, b \in \mathbb{Q}^m$.

Given a simplicial polytope P in $\mathbb{R}^d$, our strategy to construct an equivalent rational polytope will be as follows:

1. Take the dual polytope P° of P.
2. Apply Theorem 2 to find a combinatorially equivalent polytope Q° to P° defined by rational inequalities.
3. The dual $Q = (Q^\circ)^\circ$ will be equivalent to P and have rational vertices.

Without loss of generality, we may assume P is d-dimensional and contains the origin in its interior.

Let us consider why this strategy is correct. Let $(\mathcal{F}, \leq)$ be the face lattice of P. The face lattice of the dual P° is $(\mathcal{F}, \geq)$, so Q has face lattice $(\mathcal{F}, \leq)$ and is equivalent to P. To establish that Q has rational vertices, we invoke the following theorem describing the dual:

Theorem 3. $P = \operatorname{conv}\{v_1, \ldots, v_n\}$ *if and only if* $P^\circ = \{x \in \mathbb{R}^d : \langle x, v_i \rangle \leq 1 \ \forall i\}$.

It remains to prove Theorem 2.

Proof. Let P be a simple polytope described by the minimal system $\{x \in \mathbb{R}^d : Ax \leq b\}$. Let $a \in \mathbb{R}^d$ be the vector of the first row of A, and suppose that the inequality $\langle a, x \rangle \leq b_1$ contains an irrational entry. We will construct a polytope $\widetilde{P} = \{x \in \mathbb{R}^d : \widetilde{A}x \leq \widetilde{b}\}$, combinatorially equivalent to P, with the inequality $\langle a, x \rangle \leq b_1$ replaced by a rational inequality $\langle \widetilde{a}, x \rangle \leq \widetilde{b}_1$.

The notation $A[I]$ denotes the submatrix of A with the rows in the index set I, and $A(I)$ the submatrix with the rows in I removed. When I is specified to contain the first row a, we write I^- for I without a.

Let F be the facet of P corresponding to $\langle a, x \rangle \leq b_1$, and let $\widetilde{F}$ be the facet of $\widetilde{P}$ for $\langle \widetilde{a}, x \rangle \leq \widetilde{b}_1$. To preserve the combinatorial structure, we must ensure three properties:

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(1) For each vertex of F, there is a corresponding vertex of $\widetilde{F}$ which is contained in the same facets of P. An invertible $d \times d$ submatrix $A[I]$ (with a in I) whose solution was not a vertex of F is also not a vertex of $\widetilde{F}$ when a is replaced by $\widetilde{a}$.

(2) The vertices of P not in F remain vertices of $\widetilde{P}$; that is, the new inequality $\langle \widetilde{a}, x\rangle \leq \widetilde{b}_1$ is valid for them. Moreover, their adjacency relations are preserved. Conversely, points which were not vertices in P do not become vertices of $\widetilde{P}$. For this to hold, an invertible $d \times d$ submatrix $A[I]$ (with a not in I) whose solution violated $\langle a, x\rangle \leq b_1$ must also be invalid for $\langle \widetilde{a}, x\rangle \leq \widetilde{b}_1$.

(3) A singular $d \times d$ submatrix $A[I]$ (with a in I) may become invertible when a is replaced by $\widetilde{a}$. Whenever this occurs, the resulting point is not a vertex of $\widetilde{P}$.

We will proceed by showing that if $\widetilde{a}$ is sufficiently close to a, and $\widetilde{b}_1$ sufficiently close to b_1, then the three properties are satisfied. Since $\mathbb{Q}$ is dense, this implies that we can choose $\widetilde{a} \in \mathbb{Q}^d, \widetilde{b}_1 \in \mathbb{Q}$. Repeating this construction for all rows with an irrational entry then produces a rational polytope $\widetilde{P}$ which is combinatorially equivalent to P.

(1) Suppose that F contains q vertices $v_1, \ldots, v_q$, and let I_k be the index set of the rows defining vertex k for $k \in \{1, \ldots, q\}$. Since P is simple, each vertex is contained in exactly d facets, so $|I_k| = d$ for every k. For a vector $u \in \mathbb{R}^d$, let $A[I_k \leftarrow u]$ denote the matrix obtained by replacing the first row a in $A[I_k]$ with the vector u, and let U_k be the set of $u \in \mathbb{R}^d$ such that $A[I_k \leftarrow u]$ is invertible. Note that U_k is an open set with $a \in U_k$. Similarly, $b[I_k \leftarrow \beta]$ is the vector $b[I_k]$ with the first entry b_1 replaced by β.

First, observe that since $A(I_k)v_k < b(I_k)$ for each k, we can find $\epsilon_k > 0$ such that if $\|x - v_k\| < \epsilon_k$, then $A(I_k)x < b(I_k)$. Set $\epsilon = \min_{1\leq k\leq q} \epsilon_k > 0$.

For each k, define the function $f_k : U_k \oplus \mathbb{R} \to \mathbb{R}^d$ by

$$f_k(u, \beta) = A[I_k \leftarrow u]^{-1} b[I_k \leftarrow \beta]$$

That is, $f_k(u, \beta)$ is the unique solution of the system $A[I_k \leftarrow u]x = b[I_k \leftarrow \beta]$. By Cramer's formula, $f_k(u, \beta)$ is a rational function in the variables $u_1, \ldots, u_d, \beta$ and therefore continuous on $U_k \oplus \mathbb{R}$. It follows that we can find $\delta_k > 0$ such that if $\|(u, \beta) - (a, b_1)\| < \delta_k$, then $\|f_k(u, \beta) - v_k\| < \epsilon$. Set $\delta^* = \min_{1\leq k\leq q} \delta_k$.

Next, let $\mathcal{I}$ denote the set of index sets I such that a is in I and $A[I]$ is invertible, but the resulting point $x_I = A[I]^{-1}b[I]$ is not a vertex of P. This implies that there is another inequality $\langle a_I, x\rangle \leq b_I$ defining P which is violated by x_I. For each I, consider the function f_I given by

$$f_I(u, \beta) = b_I - \langle a_I, A[I \leftarrow u]^{-1} b[I \leftarrow \beta]\rangle$$

This is again a rational function and satisfies $f_I(a, b_1) < 0$, so we can find $\delta_I > 0$ such that if $\|(u, \beta) - (a, b_1)\| < \delta_I$, then $f_I(u, \beta) < 0$. Let $\delta^{**} = \min_{I\in\mathcal{I}} \delta_I$.

Let $\delta^{(1)} = \min\{\delta^*, \delta^{**}\}$. We have established that if $\|(\widetilde{a}, \widetilde{b}_1) - (a, b_1)\| < \delta^{(1)}$, then property (1) is satisfied.

(2) Next, let $A[I]$ be any invertible $d \times d$ submatrix of A which excludes the row a, and define

$$v_I = A[I]^{-1} b[I]$$

Consider the function $g_I : \mathbb{R}^d \oplus \mathbb{R} \to \mathbb{R}$ defined by

$$g_I(u, \beta) = \beta - \langle u, v_I\rangle$$

The function g_I is clearly continuous for every I. Thus,for each g_I, we may find $\delta_I > 0$ such that if $\|(u, \beta) - (a, b_1)\| < \delta_I$, then $\operatorname{sgn}(g_I(u, \beta)) = \operatorname{sgn}(g_I(a, b_1))$.

Let $\delta^{(2)} = \min_I \delta_I$. This establishes that if $\|(\widetilde{a}, \widetilde{b}_1) - (a, b_1)\| < \delta^{(2)}$, then property (2) is satisfied.

(3) Property (3) is slightly more involved. Let $\mathcal{J}$ denote the set of index sets J such that $|J| = d$, J includes the row a, and $A[J]$ is singular. Since P is a polytope and therefore bounded, the face F_J (if it exists) induced by J^- is also bounded. Since $a \in \operatorname{Col}(A[J^-]^T)$, F_J must be bounded by the inequalities $A(1)x \leq b(1)$. Hence, there exists R such that if $x \in F_J$ for any $J \in \mathcal{J}$, then $\|x\| < R$.

For $r \leq d$ and $\beta \in \mathbb{R}$, let $A_J(r,\beta)$ be the matrix obtained by replacing the r-th column of $A[J]$ by $b[J \leftarrow \beta]$. Consider the function h given by

$$h(\beta) = \prod_{J \in \mathcal{J}} \prod_{r=1}^{d} \det(A_J(r,\beta))$$

h is a polynomial in one variable, not uniformly zero, and therefore $h^{-1}(0)$ is a finite set. It follows that we can find β, arbitrarily close to b_1, such that for some $\eta > 0$, we have $|\det(A_J(r,\beta))| \geq \eta$ for every r, J.

Let us fix such a β, and take b' to be obtained from b by replacing b_1 with β. Now, let $A_J(u,r,b')$ denote the matrix obtained by replacing the first row of $A[J]$ by u, and then replacing the r-th column by b'. By Cramer's formula, the r-th entry of the solution x_J to $A[J \leftarrow u]x = b'[J]$ is given by

$$\frac{\det(A_J(u,r,b'))}{\det(A[J \leftarrow u])}$$

Our construction of b' implies that $|\det(A_J(u,r,b'))| \geq \frac{\eta}{2}$ for all u sufficiently close to a, whereas $\det(A[J \leftarrow u])$ tends to 0 as $A[J]$ is singular. Thus, we can find $\delta^{(3)}(\beta) > 0$ such that if $\|u - a\| < \delta^{(3)}(\beta)$, then $\|x_J\| > R$ for every J.

Since $\|x\| < R$ for all x contained in any F_J, and the point x_J is clearly in F_J, we deduce that x_J is not in $\widetilde{P}$. Thus, by choosing $\widetilde{b}_1$ sufficiently close to b_1 and then taking $\|\widetilde{a} - a\| < \delta^{(3)}(\widetilde{b}_1)$, property (3) is satisfied.

Choose $\widetilde{b}_1 \in \mathbb{Q}$ so that $h(\widetilde{b}_1) \neq 0$ and $|\widetilde{b}_1 - b_1| < \frac{1}{2}\min\{\delta^{(1)}, \delta^{(2)}\}$. We can then select $\widetilde{a} \in \mathbb{Q}^d$ so that $\|\widetilde{a} - a\| < \frac{1}{2}\min\{\delta^{(3)}(\widetilde{b}_1), \delta^{(1)}, \delta^{(2)}\}$. With this choice, the three properties are satisfied. Thus, replacing the first inequality $\langle a, x\rangle \leq b_1$ by $\langle \widetilde{a}, x\rangle \leq \widetilde{b}_1$ yields an equivalent polytope $\widetilde{A}x \leq \widetilde{b}$. $\square$

REFERENCES

[1] Günter M. Ziegler. *Lectures on polytopes.* Graduate Texts in Mathematics. Springer, New York, 1995.