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A CLASS OF STOCHASTIC GAMES AND MOVING FREE BOUNDARY PROBLEMS*

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Abstract. In this paper we propose and analyze a class of N-player stochastic games that include finite fuel stochastic games as a special case. We first derive sufficient conditions for the Nash equilibrium (NE) in the form of a verification theorem. The associated quasi-variational-inequalities include an essential game component regarding the interactions among players, which may be interpreted as the analytical representation of the conditional optimality for NEs. The derivation of NEs involves solving first a multidimensional free boundary problem and then a Skorokhod problem. Finally, we present an intriguing connection between these NE strategies and *controlled* rank-dependent stochastic differential equations.

Key words. finite fuel problem, free boundary problem, Markovian Nash equilibrium, N-player games, rank-dependent SDEs, reflected Brownian motion, Skorokhod problem

AMS subject classifications. 60H10, 60J60, 93E20

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1. Introduction.

Recently there has been renewed interest in N-player nonzero-sum stochastic games, inspired by the rapid growth in the theory of mean field games (MFGs) led by the pioneering work of [21, 29, 30, 31]. In this paper, we formulate and analyze a class of stochastic N-player games that originated from the classic finite fuel problem. There are many reasons to consider this type of game. First, the finite fuel problem [6, 7, 24] is one of the landmarks in stochastic control theory, and therefore mathematically a game formulation is natural. Second, in addition to the interest for stochastic control theory [3, 9, 37], its simple yet insightful solution structures have had a wide range of applications including economics and finance [8, 10, 33], operations research and management [17, 28], and queuing theory [26]. Third, prior success in analyzing its stochastic game counterpart has been restricted to the special case of two-player games [11, 19, 20, 25, 27, 34] or without the fuel constraint [12, 18].

In this paper, we will analyze a class of N-player stochastic games that include the finite fuel stochastic game as a special case. There are N players whose dynamics are governed by an N-dimensional controlled diffusion process with controls of finite variation. Each player has access to some or all of M types of resources. Players interact through their objective functions, as well as their shared resources, which are the "fuels" of their controls. The accessibility of these resources to the players and how these resources are consumed by their respective players are governed by a matrix $\boldsymbol{A} := (a_{ij})_{i,j} \in \mathbb{R}^{N\times M}$. For instance, when $M = 1$ and $\boldsymbol{A} = [1, 1, \ldots, 1]^T \in \mathbb{R}^{N\times 1}$, this

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game ($\boldsymbol{C_p}$) corresponds to the N-player finite fuel game where the N players share a fixed amount of the same resource. When $M = N$ and $\boldsymbol{A} = \boldsymbol{I_N}$, this is an N-player game ($\boldsymbol{C_d}$) where each player has her individual fixed amount of resource. In general, this matrix $\boldsymbol{A}$ describes the network structure of the N-player game. The goal for player i in the game is to minimize her cost function over appropriate admissible game strategies, which are specified in section 2. Note that this N-player game cannot be simply analyzed with an MFG approach as the network structure would collapse if an aggregation approach was applied.

We will analyze the NEs of this stochastic game. We first derive sufficient conditions for the NE policy in the form of a verification theorem (Theorem 3.1), which reveals an essential game element regarding the interactions among players. This is the Hamilton–Jacobi–Bellman (HJB) representation of the conditional optimality for NE in a stochastic game. To understand the structural properties of the NEs, we proceed further to analyze this stochastic game in terms of the game values, the NE strategies, and the controlled dynamics. Mathematically, the analysis involves solving first a multidimensional free boundary problem and then a Skorokhod problem with a *moving* boundary. The boundary is "moving" in that it moves in response to both changes of the system and controls of other players. The analytical solution is derived by first exploring the two special games $\boldsymbol{C_p}$ and $\boldsymbol{C_d}$. Analyzing these two types of games provides key insights into the solution structure of the general game. Finally, we reformulate the NE strategies in the form of controlled rank-dependent stochastic differential equations (SDEs) and compare game values between games $\boldsymbol{C_p}$ and $\boldsymbol{C_d}$.

Main contributions. (i) In the verification theorem for N-player games, we obtain the form of the HJB equations for general stochastic games with singular controls. Unlike all previous analysis that focused on two-player games, we show that in addition to the standard HJBs that correspond to stochastic control problems, there is an essential term that is unique to stochastic games. This term represents the interactions among players, especially the ones who are active and those who are waiting. This critical term was hidden in two-player stochastic games and was previously (mis)understood as a regularity condition.

(ii) The structural difference between games and control problems is further revealed in the explicit solution to the NEs for N-player games. In a control problem, a free boundary depends on the state of the system; in stochastic games, however, the "face" of the boundary moves based on the action of herself and interaction among players in the game (Figure 3). Note that this free boundary for stochastic games with an infinite time horizon *moves* in a different sense from the one in [9] for finite time control problems where the boundary is time dependent. Rather it moves due to changes of the system and the competition in the game.

(iii) This difference is further highlighted in the framework of controlled rank-dependent SDEs. To the best of our knowledge, this is the first time a stochastic game is explicitly connected with rank-dependent SDEs in a more general form, which leads to a fresh class of yet-to-be studied SDEs (section 7.2).

(iv) We recast the controlled dynamics of the game solution in the framework of *controlled* rank-dependent SDEs. Compared with the well-known rank-dependent SDEs, rank-dependent SDEs with an additional control component are new. We establish the existence of the solution by directly constructing a reflected diffusion process. (See section 7.2 for further discussions.)

(v) Finally, stochastic games considered in this paper are resource allocation games. Resource allocation problems have a wide range of applications including inventory management, resource allocation, cloud computing, smart power grid control, and multimedia wireless networks [14, 15, 32, 36]. However, the existing literature has been unsuccessful in analyzing the resource allocation problem in the setting of stochastic games. Besides the technical contributions, our analysis provides a useful economic insight: in a stochastic game of resource allocations, sharing has a lower cost than dividing, and pooling yields the lowest cost for each player.

Related work. There are a number of papers on non-zero-sum two-player games with singular controls. By treating one player as a controller and the other as a stopper, Karatzas and Li [25] analyze the existence of an NE for the game using a BSDE approach. Hernandez-Hernandez, Simon, and Zervos [20] study the smoothness of the value function and show that the optimal strategy may not be unique when the controller enjoys a first-move advantage. Kwon and Zhang [27] investigate a game of irreversible investment with singular controls and strategic exit. They characterize a class of market perfect equilibria and identify a set of conditions under which the outcome of the game may be unique despite the multiplicity of the equilibria. De Angelis and Ferrari [11] establish the connection between singular controls and optimal stopping times for a non-zero-sum two-player game. Mannucci [34] and Hamadène and Mu [19] consider the fuel follower problem in a finite-time horizon with a bounded velocity and establish via different techniques the existence of an NE of the two-player game. Very recently, [18] compared the N-player game to the MFG for the fuel follower problem. All these works are without the fuel constraint and are essentially built on one-dimensional stochastic control problems. Furthermore, except for [18], all of these papers are restricted to the case of $N = 2$. To the best of our knowledge, our work is the first to complete the mathematical analysis on an N-player stochastic game based on an original two-dimensional control problem.

In our work the controlled dynamics are recast as *controlled* rank-dependent SDEs. Rank-dependent SDEs without controls arise in the "Up the River" problem [1] and in stochastic portfolio theory [13], including the well-studied *Atlas model* [4, 22].

Notation and organization. Throughout the paper, we denote vectors/matrices by bold case letters, e.g., $\boldsymbol{x}$ and $\boldsymbol{X}$. Denote $\boldsymbol{x}^T$ as the transpose of a real vector $\boldsymbol{x}$. For a vector $\boldsymbol{x}$, $\|\boldsymbol{x}\|$ denotes its l_2 norm. For a matrix $\boldsymbol{X}$, $\|\boldsymbol{X}\|$ denotes its spectral norm.

The paper is organized as follows. Section 2 presents the mathematical formulation of the N-player game. Section 3 provides a verification theorem for sufficient conditions of the NE of the game and the existence of the Skorokhod problem for NE strategies. Section 4 studies game $\boldsymbol{C_p}$ and section 5 studies game $\boldsymbol{C_d}$. With the insight from these two games, section 6 analyzes the general N-player game $\boldsymbol{C}$. Section 7 compares games $\boldsymbol{C_p}$, $\boldsymbol{C_d}$, and $\boldsymbol{C}$, discusses the game values and their economic implications, and unifies their corresponding controlled dynamics in the framework of the controlled rank-dependent SDEs.

We provide some technical proofs in the online supplementary material, which can be found at https://arxiv.org/pdf/1809.03459.pdf.

2. Problem setup. Now we present the mathematical formulation for the stochastic N-player game.

Controlled dynamics. Let $(X_t^i)_{t\geq 0}$ be the position of player $i, 1 \leq i \leq N$. In the absence of controls, $\boldsymbol{X}_t = (X_t^1, \ldots, X_t^N)$ is governed by the SDE:

$$d\boldsymbol{X}_t = \boldsymbol{b}(\boldsymbol{X}_t)dt + \boldsymbol{\sigma}(\boldsymbol{X}_t)d\boldsymbol{B}_t, \quad \boldsymbol{X}_{0-} = (x^1, \ldots, x^N), \tag{2.1}$$

where $\boldsymbol{B} := (B^1, \ldots, B^N)$ is a standard N-dimensional Brownian motion in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the drift $\boldsymbol{b}(\cdot) := (b_1(\cdot), \ldots, b_N(\cdot))$ and the covariance matrix $\boldsymbol{\sigma}(\cdot) := (\sigma_{ij}(\cdot))_{1\leq i,j\leq N}$. As will be explained later in section 3.3, we consider a weak formulation of the stochastic game. To ensure the existence and the uniqueness of the SDE, $\boldsymbol{b}(\cdot)$ and $\boldsymbol{\sigma}(\cdot)$ are assumed to satisfy this condition:

H1. $\boldsymbol{b}(\cdot)$ and $\boldsymbol{\sigma}(\cdot)$ are bounded and continuous, and $\boldsymbol{\sigma}(\cdot)$ is uniformly elliptic, i.e., there exists $\alpha > 0$ such that $\xi^T \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\sigma}^\top(\boldsymbol{x})\xi \geq \alpha|\xi|^2$ for all $\boldsymbol{x} \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$.

Assumption H1 ensures the existence of a *weak solution* to (2.1) [38]. Here and throughout the rest of the paper, the infinitesimal generator $\mathcal{L}$ is

$$\mathcal{L} := \sum_i b_i(\boldsymbol{x})\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j}(\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\sigma}(\boldsymbol{x})^T)_{i,j}\frac{\partial^2}{\partial x^i \partial x^j}, \tag{2.2}$$

where $\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\sigma}(\boldsymbol{x})^T$ is assumed to be positive-definite for every $\boldsymbol{x} \in \mathbb{R}^N$.

If a control is applied to X_t^i, then X_t^i evolves as

$$dX_t^i = b_i(\boldsymbol{X}_{t-})dt + \boldsymbol{\sigma}_i(\boldsymbol{X}_{t-})d\boldsymbol{B}_t + d\xi_t^{i+} - d\xi_t^{i-}, \quad X_{0-}^i = x^i, \tag{2.3}$$

where $\boldsymbol{\sigma}_i$ is the ith row of the covariance matrix $\boldsymbol{\sigma}$. Here the control (ξ^{i+}, ξ^{i-}) is a pair of nondecreasing and càdlàg processes. In other words, (ξ^{i+}, ξ^{i-}) is the minimum decomposition of the finite variation process ξ^i such that $\xi^i := \xi^{i+} - \xi^{i-}$.

Game objective. The game is for player i to minimize, for all (ξ^{i+}, ξ^{i-}) in an appropriate admissible control set, the following objective function:

$$\mathbb{E}\int_0^\infty e^{-\alpha t} h^i\left(X_t^1, \ldots, X_t^N\right)dt. \tag{2.4}$$

Here $\alpha > 0$ is a constant discount factor. In this game, players interact through their respective objective functions $h^i(\boldsymbol{x}) : \mathbb{R}^N \to \mathbb{R}^+$.

H2. $h^i(\boldsymbol{x})$ is twice differentiable, with $k \leq ||\nabla^2 h^i(\boldsymbol{x})|| \leq K$ for some $K > k > 0$.

For example, $h^i(\boldsymbol{x}) = h(x^i - \frac{\sum_{j=1}^N x^j}{N})$ with $h(\cdot) \geq 0$ is a distance function between the position of player i and the center of all players.

Note that in the objective function (2.4), there is no cost of control. With this formulation, the explicit solution structure of the NE for game (2.4) is neat and insightful. It is entirely possible to consider an N-player game with additional cost of control. For instance, one might study the game formulation of [24] with a proportional cost of control. We conjecture that the solution structure would be similar although the analysis will be more involved.

Admissible control policies. Denote $\check{\xi}_t^i$ as the cumulative resources/controls consumed by player i up to time t. When ξ_t^i is of finite variation, then there is a unique decomposition such that $\xi_t^i := \xi_t^{i+} - \xi_t^{i-}$, hence $\check{\xi}_t^i := \xi_t^{i+} + \xi_t^{i-}$. Here ξ^{i+} and ξ^{i-} are nondecreasing càdlàg processes which can be further decomposed in a differential form, $d\xi_t^{i\pm} = d(\xi_t^{i\pm})^c + \Delta\xi_t^{i\pm}$, where $d(\xi_t^{i\pm})^c$ is the continuous component and $\Delta\xi_t^{i\pm} := \xi_t^{i\pm} - \xi_{t-}^{i\pm}$ is the jump component of $d\xi_t^{i\pm}$. Equivalently, we can write $\xi_t^{i\pm} = (\xi_t^{i\pm})^c + \sum_{s\leq t}\Delta\xi_s^{i\pm}$.

Meanwhile, we consider a *weak formulation* of the stochastic game. (See [39, Chapter 2, section 4.2] and [16, section 5] for more discussions on weak formulations of stochastic control problems). That is, $(\boldsymbol{B}_t,\, t \geq 0)$ is an N-dimensional Brownian motion with some filtration $(\mathcal{F}_t,\, t \geq 0)$, and the admissible control set $\mathcal{S}_N(\boldsymbol{x}, \boldsymbol{y})$ for the N-player game is

$$\mathcal{S}_N(\boldsymbol{x},\boldsymbol{y}) := \Bigg\{\boldsymbol{\xi} \,:\, \xi^i \in \mathcal{U}_N^i \text{ for } 1 \leq i \leq N,\ \sum_{i=1}^N \int_0^\infty \frac{a_{ij}Y_{t-}^j}{\sum_{k=1}^M a_{ik}Y_{t-}^k} d\check{\xi}_t^i \leq y^j, 1 \leq j \leq M,$$

$$\mathbb{P}\left(\Delta\xi_t^i \Delta\xi_t^k \neq 0\right) = 0 \text{ for all } t \geq 0 \text{ and } i \neq k\Bigg\}, \quad \text{where} \tag{2.5}$$

$\mathcal{U}_N^i := \{(\xi^+, \xi^-) : \xi^+$ and ξ^- are $\mathcal{F}_t$-progressively measurable, càdlàg, nondecreasing,

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha t} d\xi_t^{\pm}\right] < \infty \text{ and } \xi_{0-}^+ = \xi_{0-}^- = 0\Big\} \quad \text{and} \tag{2.6}$$

$$Y_t^j = y^j - \sum_{i=1}^N \int_0^t \frac{a_{ij} Y_{s-}^j}{\sum_{k=1}^M a_{ik} Y_{s-}^k} d\check{\xi}_s^i \in \mathbb{R}_+ \quad \text{and} \quad Y_{0-}^j = y^j, \tag{2.7}$$

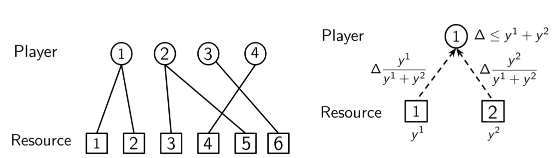
with $a_{ij} = 0$ or 1 for $1 \le i \le N$ and $1 \le j \le M$, $\sum_{j=1}^M a_{ij} > 0$ for all $i = 1, \ldots, N$, and $\sum_{i=1}^N a_{ij} > 0$ for all $j = 1, \ldots, M$.

Here are intuitions for the admissible control set $\mathcal{S}_N(\boldsymbol{x}, \boldsymbol{y})$. In this game, each player i will make decisions based on the current positions of all players and the available resources. In addition to this adaptedness constraint, the admissible control set $\mathcal{S}_N(\boldsymbol{x}, \boldsymbol{y})$ specifies the resource allocation policy for each player. For M different types of resources, define $\boldsymbol{A} := (a_{ij})_{i,j} \in \mathbb{R}^{N \times M}$ to be the *adjacent matrix* with $a_{ij} = 0$ or 1. Then $\boldsymbol{A}$ describes the relationship between the players and the types of available resources, with $a_{ij} = 1$ meaning that resource of type j is available to player i, and $a_{ij} = 0$ meaning that a resource of type j is inaccessible to player i. The condition $\sum_{j=1}^M a_{ij} > 0$ for all $1 \le i \le N$ implies that each player i has access to at least one resource, and the condition $\sum_{i=1}^N a_{ij} > 0$ for all $1 \le j \le M$ indicates that each resource j is available to at least one player. When player i exercises control, she consume resources proportionally to all the resources available to her. She stops consuming once all the available resources hit level zero. This results in the form of the integrand in the expression of (2.7). Note that the denominator is no smaller than the numerator, hence the integrand is well-defined with the convention $\frac{0}{0} = 0$.

Take an example of $N = 4$, $M = 6$, with the matrix $\boldsymbol{A}$ defined as in Figure 1. The resource allocation policy is illustrated in Figure 1(b), with the amount of available resources y^1 and y^2 of type one and two, respectively. When player one wishes to apply controls of amount Δ, say, $\Delta \le y^1 + y^2$, she will take $\Delta \frac{y^1}{y^1+y^2}$ from resource one and $\Delta \frac{y^2}{y^1+y^2}$ from resource two. Finally, the condition $\mathbb{P}(\Delta\xi_t^i \Delta\xi_t^k \neq 0) = 0$ for all $t \ge 0$ and $i \neq k$ excludes the possibility of simultaneous jumps of any two out of N players, which facilitates designing feasible control policies when controls involve jumps. This condition is not a restriction and instead should be interpreted as a *regularization.* See also [5, 18, 27].

Game formulation and game criterion. Let $\boldsymbol{\xi} := (\xi^1, \ldots, \xi^N)$ be the controls from the players. Let $\boldsymbol{x} := (x^1, \ldots, x^N)$ and $\boldsymbol{y} := (y^1, \ldots, y^M)$. Then the stochastic game is for each player i to minimize

$$\boldsymbol{A} = \begin{bmatrix} 1,1,0,0,0,0 \\ 0,0,1,0,1,0 \\ 0,0,0,0,0,1 \\ 0,0,0,1,0,0 \end{bmatrix}$$



(a) Relationship

(b) Resource allocation policy

FIG. 1. *Example of adjacent matrix* $\boldsymbol{A}$*, relationship between the players, and resources when* $N = 4$ *and* $M = 6$.

$$(2.8)\qquad J^i(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi}) := \mathbb{E}\int_0^\infty e^{-\alpha t}h^i(\boldsymbol{X}_t)dt,$$

subject to the dynamics in (2.3) and (2.7) with the constraint in (2.5).

There are two special games of particular interest. One is a game where all players pool their resources such that $\sum_{i=1}^N \check{\xi}^i_\infty \le y < \infty$. When $N=1$, this is a single player game corresponding to the finite fuel control problem, which is well studied in [7, 24]. We call this game a pooling game $\boldsymbol{C_p}$. Clearly in terms of the adjacent matrix $\boldsymbol{A}$, this corresponds to $M=1$, and $\boldsymbol{A}=[1,1,\dots,1]^T \in \mathbb{R}^{N\times 1}$. Another is a game where players divide the resource up front such that $\check{\xi}^i_\infty \le y^i$, with y^i the total resources that player i can exercise. This game is called $\boldsymbol{C_d}$, with $M=N$ and $\boldsymbol{A}=\boldsymbol{I_N}$. Finally, a game with a general matrix $\boldsymbol{A}$ is denoted as game $\boldsymbol{C}$.

We will analyze the N-player game under the criterion of NE. Recall the definition of NE of N-player games.

Definition 2.1. *A tuple of admissible controls $\boldsymbol{\xi}^* := (\xi^{1*},\dots\xi^{N*})$ is a NE of the N-player game* (2.8) *if for each $\xi^i \in \mathcal{U}_N^i$ such that $(\boldsymbol{\xi}^{-i*},\xi^i)\in\mathcal{S}_N(\boldsymbol{x},\boldsymbol{y})$,*

$$J^i(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi}^*) \le J^i\left(\boldsymbol{x},\boldsymbol{y};\left(\boldsymbol{\xi}^{-i*},\xi^i\right)\right),$$

where $\boldsymbol{\xi}^{-i*} = (\xi^{1*},\dots,\xi^{i-1*},\xi^{i+1*},\dots,\xi^{N*})$ and $(\boldsymbol{\xi}^{-i*},\xi^i) = (\xi^{1*},\dots,\xi^{i-1*},\xi^i,\xi^{i+1*},\dots,\xi^{N*})$. Controls that give NEs are called the Nash equilibrium points (NEPs). The associated value function $J^i(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi}^*)$ is called the game value for player i $(1\le i\le N)$.

3. NE game solution: Verification theorem and Skorokhod problem.

In this section, we present general strategies to get the NE solution. First, we derive heuristically the quasi-variational inequalities (QVIs) for the value function (section 3.1), which is then used for deriving sufficient conditions of an NEP via a verification theorem (section 3.2). We emphasize that both the QVIs in section 3.1 and the verification theorem in section 3.2 hold for general diffusion processes given in (2.3). For explicitness, we assume further that

H1′. $b_i = 0, \qquad i=1,2,\dots,N, \qquad$ and $\qquad \boldsymbol{\sigma}=\boldsymbol{I}_N$.

Moreover, we assume that $h^i(\boldsymbol{x}) := h(x^i - \frac{1}{N}\sum_{j=1}^N x^j)$, such that

H2′. h is symmetric, $h(0)\ge 0$, h'' is nonincreasing on $\mathbb{R}_+$ and $k\le h''\le K$ for some $0<k<K$.

These additional conditions are only used to facilitate the construction of the NEP, as well as solving the corresponding Skorokhod problem presented in section 3.3.

3.1. Quasi-variational inequalities.

We first derive heuristically the associated QVIs of game value under the notion NE (see Definition 2.1) for game (2.8). The key idea is to utilize the conditional optimality condition introduced in Definition 2.1. Namely, player i solves a single agent optimal control problem with optimal solution ξ^{i*} when other agents are applying $\boldsymbol{\xi}^{-i*}$. To start, we define the following partition of $\mathbb{R}^N\times\mathbb{R}_+^M$. Denote $\mathcal{A}_i\subseteq\mathbb{R}^N\times\mathbb{R}_+^M$ as the ith player's action region and $\mathcal{W}_i := (\mathbb{R}^N\times\mathbb{R}_+^M)\backslash\mathcal{A}_i$ as her waiting region. Let $\mathcal{A}^{-i} := \cup_{j\ne i}\mathcal{A}_j$ and $\mathcal{W}_{-i} := \cap_{j\ne i}\mathcal{W}_j$. Then the players' actions are as follows: player i controls if and only if the process $(\boldsymbol{X}_t,\boldsymbol{Y}_t)$ enters $\mathcal{A}_i$. This partition is usually defined via the QVIs and is also part of the solution to be derived. Next, define the intervene operator Γ as

$$(3.1)\qquad \Gamma_j v^i(\boldsymbol{x},\boldsymbol{y}) = \sum_{k=1}^M \frac{a_{jk}y^k}{\sum_{s=1}^M a_{js}y^s} v^i_{y^k}(\boldsymbol{x},\boldsymbol{y})$$

for $(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}_+^M$ and $i,j = 1,2,\ldots,N$. Here $v^i_{y^k} := \frac{\partial v^i}{\partial y^k}$ ($i = 1,2,\ldots,N$ and $k = 1,2,\ldots,M$). Suppose player j takes a possibly suboptimal action $\Delta\xi^{j,+} > 0$; then by the resource allocation policy (2.7), for player i,

$$v^i(\boldsymbol{x},\boldsymbol{y}) \le v^i\left(\boldsymbol{x}^{-j}, x^j + \Delta\xi^{j,+}, \boldsymbol{y} - \left(\frac{a_{j1}y^1}{\sum_{k=1}^M a_{jk}y^k}, \ldots, \frac{a_{jM}y^M}{\sum_{k=1}^M a_{jk}y^k}\right)\Delta\xi^{j,+}\right). \tag{3.2}$$

By letting $\Delta\xi^{j,+} \to 0$, we have $0 \le -\Gamma_j v^i(\boldsymbol{x},\boldsymbol{y}) + v^i_{x^j}(\boldsymbol{x},\boldsymbol{y})$.

Next, we provide the heuristics for deriving the QVIs. Let $\Delta\xi^i := \Delta\xi^i(\boldsymbol{x},\boldsymbol{y})$ be the control of player i with joint state position $(\boldsymbol{x},\boldsymbol{y})$. When $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{W}_{-i}$, we have $\Delta\xi^j = 0$ for $j \ne i$. Thus the game for player i becomes a classical control problem with three choices: $\Delta\xi^i = 0$, $\Delta\xi^{i,+} > 0$, and $\Delta\xi^{i,-} > 0$. The first case $\Delta\xi^i = 0$ implies, by simple stochastic calculus, $-\alpha v^i + h^i(\boldsymbol{x}) + \mathcal{L}v^i \ge 0$. Similarly, the second case $\Delta\xi^{i,+} > 0$ corresponds to $-\Gamma_i v^i + v^i_{x^i} \ge 0$ and the third case $\Delta\xi^{i,-} > 0$ corresponds to $-\Gamma_i v^i - v^i_{x^i} \ge 0$. Since one of the three choices will be optimal, one of the inequalities will be an equation. That is, for $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{W}_{-i}$,

$$\min\left\{-\alpha v^i + h^i(\boldsymbol{x}) + \mathcal{L}v^i, -\Gamma_i v^i + v^i_{x^i}, -\Gamma_i v^i - v^i_{x^i}\right\} = 0. \tag{3.3}$$

When $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_j$, player j will control with $(\Delta\xi^{j,+}, \Delta\xi^{j,-}) \ne 0$. Therefore,

$$v^j(\boldsymbol{x},\boldsymbol{y}) \le v^j\left(\boldsymbol{x}^{-j}, x^j + \Delta\xi^{j,+}, \boldsymbol{y} - \left(\frac{a_{j1}y^1}{\sum_{k=1}^M a_{jk}y^k}, \ldots, \frac{a_{jM}y^M}{\sum_{k=1}^M a_{jk}y^k}\right)\Delta\xi^{j,+}\right), \tag{3.4}$$

$$v^j(\boldsymbol{x},\boldsymbol{y}) \le v^j\left(\boldsymbol{x}^{-j}, x^j - \Delta\xi^{j,-}, \boldsymbol{y} - \left(\frac{a_{j1}y^1}{\sum_{k=1}^M a_{jk}y^k}, \ldots, \frac{a_{jM}y^M}{\sum_{k=1}^M a_{jk}y^k}\right)\Delta\xi^{j,-}\right), \tag{3.5}$$

and one of the inequalities in (3.4)–(3.5) will be an equality. This leads to the following:

$$\min\left\{-\Gamma_j v^j + v^j_{x^j}, -\Gamma_j v^j - v^j_{x^j}\right\} = 0. \tag{3.6}$$

For player $i \ne j$, we should have $v^i(\boldsymbol{x},\boldsymbol{y}) = v^i(\boldsymbol{x}^{-j}, x^j + \Delta\xi^{j,+}, \boldsymbol{y} - (\frac{a_{j1}y^1}{\sum_{k=1}^M a_{jk}y^k}, \ldots, \frac{a_{jM}y^M}{\sum_{k=1}^M a_{jk}y^k})\Delta\xi^{j,+})$ when $\Delta\xi^{j,+} > 0$ is optimal for player j, and $v^i(\boldsymbol{x},\boldsymbol{y}) = v^i(\boldsymbol{x}^{-j}, x^j - \Delta\xi^{j,-}, \boldsymbol{y} - (\frac{a_{j1}y^1}{\sum_{k=1}^M a_{jk}y^k}, \ldots, \frac{a_{jM}y^M}{\sum_{k=1}^M a_{jk}y^k})\Delta\xi^{j,-})$ when $\Delta\xi^{j,-} > 0$ is optimal for player j. This holds due to the "no simultaneous jump" condition (2.5). This implies that player i has no incentive to jump when player j jumps. Thus,

$$\begin{cases} -\Gamma_j v^i + v^i_{x^j} = 0, \text{ on } \{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}_+^M \mid -\Gamma_j v^j + v^j_{x^j} = 0\}, \\ -\Gamma_j v^i - v^i_{x^j} = 0, \text{ on } \{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}_+^M \mid -\Gamma_j v^j - v^j_{x^j} = 0\}. \end{cases} \tag{3.7}$$

Note that by letting $\Delta\xi^{i,\pm} \to 0$, (3.3), (3.6), and (3.7) describe the behavior in $\overline{\mathcal{W}}_i$ and near boundary $\partial\mathcal{W}_i$. Moreover, we can show that (3.3), (3.6), and (3.7) are consistent with the jump behaviors in $\mathcal{A}_i$. To see this, $-\sum_{j=1}^M \frac{a_{ij}y^j}{\sum_{k=1}^M a_{ik}y^k} v^i_{y^j} \pm v^i_{x^i} = 0$ has a linear solution $v^i(\boldsymbol{x},\boldsymbol{y}) = a(\pm x_i + \sum_{j=1}^M a_{ij}y^j) + b$ for some $a, b \in \mathbb{R}$. And it is easy to check that if $\sum_{k=1}^M a_{ik}y^k \ge \Delta > 0$, we have

$$\frac{a_{ij}y^j - \frac{a_{ij}y^j}{\sum_{k=1}^M a_{ik}y^k}\Delta}{\sum_{k=1}^M a_{ik}y^k - \Delta} = \frac{a_{ij}y^j}{\sum_{k=1}^M a_{ik}y^k}$$

holds. This means that the allocation policy (jump direction) outside the waiting region is linear. Hence, the noninfinitesimal jump also satisfies (3.3) in $\mathcal{A}_i$. The consistency property also holds for (3.7). In summary, we have the following QVIs:

$$\min\left\{-\alpha v^i + h^i(\boldsymbol{x}) + \mathcal{L}v^i, -\Gamma_i v^i + v^i_{x^i}, -\Gamma_i v^i - v^i_{x^i}\right\} = 0,$$
$$\text{on } \cap_{j\neq i}\left\{\left\{-\Gamma_j v^j + v^j_{x^j} > 0\right\} \cap \left\{-\Gamma_j v^j - v^j_{x^j} > 0\right\}\right\}, \tag{3.8a}$$
$$-\Gamma_j v^i + v^i_{x^j} = 0 \qquad \text{on } \{-\Gamma_j v^j + v^j_{x^j} = 0\}, \tag{3.8b}$$
$$-\Gamma_j v^i - v^i_{x^j} = 0 \qquad \text{on } \{-\Gamma_j v^j - v^j_{x^j} = 0\}. \tag{3.8c}$$

The above conditions are consistent with the conditional optimality of NE for each player and describe interactions between the player in control and those who are not; these conditions ensure that all players control optimally and push sequentially the underlying dynamics until reaching the common waiting region.

3.2. Verification theorem. Next, we present a verification theorem which gives sufficient conditions of an NEP. Given functions v^i (with sufficient regularity), define the action and waiting regions ($\mathcal{A}_i$ and $\mathcal{W}_i$) in terms of v^i ($1 \le i \le N$) as the following:

$$\mathcal{A}_i = \mathcal{A}_i^+ \cup \mathcal{A}_i^-, \tag{3.9}$$

where $\mathcal{A}_i^+ := \{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}^M_+ \mid -\Gamma_i v^i - v^i_{x^i} = 0\}$ and $\mathcal{A}_i^- := \{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}^M_+ \mid -\Gamma_i v^i + v^i_{x^i} = 0\}$. Moreover, $\mathcal{W}_i = (\mathbb{R}^N \times \mathbb{R}^M_+) \setminus \mathcal{A}_i$ and $\mathcal{W}_{-i} = \cap_{j\neq i}\mathcal{W}_j$.

THEOREM 3.1 (verification theorem). *Assume* H1–H2 *hold and further assume $\mathcal{A}_j \cap \mathcal{A}_i = \emptyset$ for all $i \neq j$, where $\mathcal{A}_i, \mathcal{W}_i$, and $\mathcal{W}_{-i}$ are defined according to* (3.9). *For each $i = 1,\ldots,N$, suppose that the* ith *player's strategy $\xi^{i*} \in \mathcal{U}^i_N$ satisfies the following conditions:*

(i) $\boldsymbol{\xi}^* := (\xi^{1*},\ldots,\xi^{N*}) \in \mathcal{S}_N(\boldsymbol{x},\boldsymbol{y})$.

(ii) *$v^i(\cdot)$ satisfies the QVIs* (3.8).

(iii) *For any $\xi^i \in \mathcal{U}^i_N$ such that $(\boldsymbol{\xi}^{-i*},\xi^i) \in \mathcal{S}_N(\boldsymbol{x},\boldsymbol{y})$, $\mathbb{P}((\boldsymbol{X}^{-i*}_t, X^i_t, \boldsymbol{Y}_t) \in \overline{\mathcal{W}_{-i}}) = 1$ for all $t \ge 0$, where $(\boldsymbol{X}^{-i*}_t, X^i_t, \boldsymbol{Y}_t)$ is under $(\boldsymbol{\xi}^{-i*},\xi^i)$.*

(iv) *$v^i(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{C}^2(\overline{\mathcal{W}_{-i}})$ and v^i is convex for all $(\boldsymbol{x},\boldsymbol{y}) \in \overline{\mathcal{W}_{-i}}$.*

(v) *$\mathbb{E}[\int_0^T e^{-2\alpha t}(v^i_{x^j}(\boldsymbol{X}^{-i*}_t, X^i_t, \boldsymbol{Y}_t))^2 dt] < \infty$ for all $T > 0$, where $(\boldsymbol{X}^{-i*}_t, X^i_t, \boldsymbol{Y}_t)$ is under $(\boldsymbol{\xi}^{-i*},\xi^i) \in \mathcal{S}_N(\boldsymbol{x},\boldsymbol{y})$ such that* (iii) *holds.*

(vi) *For any $(\boldsymbol{X}^{-i*}_t, X^i_t, \boldsymbol{Y}_t)$ under $(\boldsymbol{\xi}^{-i*},\xi^i) \in \mathcal{S}_N(\boldsymbol{x},\boldsymbol{y})$ such that* (iii) *holds, $v^i(\boldsymbol{x},\boldsymbol{y})$ satisfies the transversality condition*

$$\limsup_{T\to\infty} e^{-\alpha T}\mathbb{E}\left[v^i\left(\boldsymbol{X}^{-i*}_t, X^i_t, \boldsymbol{Y}_t\right)\right] = 0. \tag{3.10}$$

(vii) *For $j \neq i$, $t \ge 0$, and $(\boldsymbol{X}^{-i*}_t, X^i_t, \boldsymbol{Y}_t)$ under $(\boldsymbol{\xi}^{-i*},\xi^i)$,*

$$\check{\xi}^{j*}_t = \int_{[0,t]} 1_{\{(\boldsymbol{X}^{-i*}_{s-}, X^i_{s-}, \boldsymbol{Y}_{s-}) \in \mathcal{A}_j\}} d\check{\xi}^{j*}_s, \tag{3.11}$$

and in addition for $(\boldsymbol{X}^*_t, \boldsymbol{Y}^*_t)$ under $\boldsymbol{\xi}^*$,

$$\check{\xi}^{i*}_t = \int_{[0,t]} 1_{\{(\boldsymbol{X}^*_{s-}, \boldsymbol{Y}^*_{s-}) \in \mathcal{A}_i\}} d\check{\xi}^{i*}_s. \tag{3.12}$$

Then $\boldsymbol{\xi}^*$ is an NEP with value function v^i, a solution to (3.8)*. That is,*

$$v^i(\boldsymbol{x},\boldsymbol{y}) \le J^i(\boldsymbol{x},\boldsymbol{y};(\boldsymbol{\xi}^{-i*},\xi^i))$$

for all $\xi \in \mathcal{U}_N^i$ such that $(\boldsymbol{\xi}^{-i*},\xi^i) \in \mathcal{S}_N$, and $v^i(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi}^*) = J^i(\boldsymbol{x},\boldsymbol{y};(\boldsymbol{\xi}^{-i*},\xi^{i*}))$.

Proof. It suffices to prove that for all $(\boldsymbol{\xi}^{-i*},\xi^i) \in \mathcal{S}_N(\boldsymbol{x},\boldsymbol{y})$, and for each $i = 1,\dots,N$, we have $J^i(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi}^*) \le J^i(\boldsymbol{x},\boldsymbol{y};(\boldsymbol{\xi}^{-i*},\xi^i))$.

Recall (2.1) and (2.7). From condition (iii), under control $(\boldsymbol{\xi}^{-i*},\xi^i) \in \mathcal{S}_N(\boldsymbol{x},\boldsymbol{y})$, $(\boldsymbol{X}_t^{-i*},X_t^i,\boldsymbol{Y}_t) \in \overline{\mathcal{W}_{-i}}$ a.s. Applying the Itô-Meyer formula [35, Theorem 21] to $e^{-\alpha t}v^i(\boldsymbol{X}_t^{-i*},X_t^i,\boldsymbol{Y}_t)$,

$$\begin{aligned}
&\mathbb{E}[e^{-\alpha T}v^i(\boldsymbol{X}_T^{-i*},X_T^i,\boldsymbol{Y}_T)] - v^i(\boldsymbol{x},\boldsymbol{y})\\
&\quad= \mathbb{E}\int_0^T e^{-\alpha t}\left(\mathcal{L}v^i - \alpha v^i\right)dt + \mathbb{E}\int_0^T e^{-\alpha t}\sum_{j=1}^N v^i_{x^j}\,dB_t^j\\
&\qquad+ \sum_{j=1,j\neq i}^N \mathbb{E}\int_{[0,T)} e^{-\alpha t}(v^i_{x^j}d\xi_t^{j*,+} - v^i_{x^j}d\xi_t^{j*,-})\\
&\qquad- \sum_{j=1,j\neq i}^N \mathbb{E}\int_{[0,T)} e^{-\alpha t}\Gamma_j v^i(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-})\left(d\xi_t^{j*,+} + d\xi_t^{j*,-}\right)\\
&\qquad+ \mathbb{E}\int_{[0,T)} e^{-\alpha t}(v^i_{x^i}d\xi_t^{i,+} - v^i_{x^i}d\xi_t^{i,-})\\
&\qquad- \mathbb{E}\int_{[0,T)} e^{-\alpha t}\Gamma_i v^i(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-})\left(d\xi_t^{i,+} + d\xi_t^{i,-}\right)\\
&\qquad+ \mathbb{E}\sum_{0\le t<T} e^{-\alpha t}\left(\Delta v^i - \sum_{j=1}^N v^i_{x^j}\Delta X_t^j - \sum_{k=1}^M v^i_{y^k}\Delta Y_t^k\right),
\end{aligned}$$

where Γ_i and Γ_j are defined in (3.1). Here $\Delta v^i := v^i(\boldsymbol{X}_t^{-i*},X_t^i,\boldsymbol{Y}_t) - v^i(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-})$, $v^i_{x^j} := v^i_{x^j}(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-})$, $v^i_{y^k} := v^i_{y^k}(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-})$, $\Delta X_t^{j*} := X_t^{j*} - X_{t-}^{j*}$, $\Delta X_t^i := X_t^i - X_{t-}^i$, and $\Delta Y_t^k := Y_t^k - Y_{t-}^k$ on the right-hand side of the above equation for $1 \le i,j \le N$ and $1 \le k \le M$. By [2, Theorem 3.2.1], condition (v) implies that the Itô integral $\int_0^T e^{-\alpha t}\sum_{j=1}^N v^i_{x^j}dB_t^j$ is a martingale. Hence, $\mathbb{E}[\int_0^T e^{-\alpha t}\sum_{j=1}^N v^i_{x^j}dB_t^j] = 0$. The convexity condition in (iv) implies $\mathbb{E}\sum_{0\le t<T} e^{-\alpha t}(\Delta v^i - \sum_{k\neq i}^N v^i_{x^k}\Delta X_t^{k*} - v^i_{x^i}\Delta X_t^i - \sum_{j=1}^M v^i_{y^j}\Delta Y_t^j) \ge 0$. Next, we have

$$\begin{aligned}
&\mathbb{E}\int_{[0,T)} e^{-\alpha t}(v^i_{x^i}d\xi_t^{i,+} - v^i_{x^i}d\xi_t^{i,-}) - \mathbb{E}\int_{[0,T)} e^{-\alpha t}\Gamma_i v^i(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-})\left(d\xi_t^{i,+} + d\xi_t^{i,-}\right)\\
&\quad= \mathbb{E}\int_{[0,T)} e^{-\alpha t}\left[v^i_{x^i}(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-}) - \Gamma_i v^i(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-})\right]d\xi_t^{i,+}\\
&\qquad+ \mathbb{E}\int_{[0,T)} e^{-\alpha t}\left[-v^i_{x^i}(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-}) - \Gamma_i v^i(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-})\right]d\xi_t^{i,-} \ge 0.
\end{aligned}$$

The last inequality holds due to conditions (ii) and (iv). More precisely, $v^i(\boldsymbol{x})$ satisfies the HJB equation (3.8a) in $\mathcal{W}_{-i}$. Along with (iv), we have the following with probability one: $v^i_{x^i}(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-}) - \Gamma_i v^i(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-}) \ge 0$ and $-v^i_{x^i}(\boldsymbol{X}_{t-}^{-i*},X_{t-}^i,\boldsymbol{Y}_{t-}) -$

$\Gamma_i v^i(\boldsymbol{X}_{t-}^{-i*}, X_{t-}^i, \boldsymbol{Y}_{t-}) \geq 0$. For each $j \neq i$, almost surely, we have $d\xi_t^{j*} \neq 0$ only when $(\boldsymbol{X}_t, \boldsymbol{Y}_t) \in \partial\mathcal{W}_{-i} \cap \partial\mathcal{A}_j$. Along with the condition (ii) and (3.8b)–(3.8c),

$$\begin{aligned}
&\mathbb{E}\int_{[0,T)} e^{-\alpha t}(v^i_{x^j}(\boldsymbol{X}_{t-}^{-i*}, X_{t-}^i, \boldsymbol{Y}_{t-})d\xi_t^{j*,+} - v^i_{x^j}(\boldsymbol{X}_{t-}^{-i*}, X_{t-}^i, \boldsymbol{Y}_t)d\xi_t^{j*,-})\\
&\quad -\mathbb{E}\int_{[0,T)} e^{-\alpha t}\Gamma_j v^i(\boldsymbol{X}_{t-}^{-i*}, X_{t-}^i, \boldsymbol{Y}_t)\left(d\xi_t^{j*,+} + d\xi_t^{j*,-}\right)\\
&\quad = \mathbb{E}\int_{[0,T)} e^{-\alpha t}\left[v^i_{x^j} - \Gamma_j v^i\right](\boldsymbol{X}_{t-}^{-i*}, X_{t-}^i, \boldsymbol{Y}_t)d\xi_t^{j*,+}\\
&\qquad + \left[-v^i_{x^j} - \Gamma_j v^i\right](\boldsymbol{X}_{t-}^{-i*}, X_{t-}^i, \boldsymbol{Y}_t)d\xi_t^{j*,-} = 0.
\end{aligned}$$

Condition (ii) also implies $\mathcal{L}v^i - \alpha v^i \geq -h$. Combining all of the above,

$$e^{-\alpha T}\mathbb{E}v^i(\boldsymbol{X}_T^{-i*}, X_T^i, \boldsymbol{Y}_T) + \mathbb{E}\int_0^T e^{-\alpha t} h\left(\boldsymbol{X}_t^{-i*}, X_t^i\right)dt \geq v^i(\boldsymbol{x}, \boldsymbol{y}). \tag{3.13}$$

By letting $T \to \infty$, (3.13) and condition (vi) lead to the desirable inequality.

The equality in (3.13) holds for $\xi^i = \xi^{i*}$ by (3.12), and $\mathbb{P}((\boldsymbol{X}_t^*, \boldsymbol{Y}_t^*) \in \overline{\cap_{i=1}^N \mathcal{W}_i}) = 1$ for all $t \geq 0$ and the no simultaneous jump condition in the admissible set (2.5), where $(\boldsymbol{X}_t^*, \boldsymbol{Y}_t^*)$ is the dynamics under $\boldsymbol{\xi}^*$. ∎

Supposing the game value v^i $(i = 1, 2, \ldots, N)$ that satisfies the verification theorem (Theorem 3.1) is given, the next step is to construct the corresponding NE strategies. This is done by solving a Skorokhod problem, discussed in the next subsection.

3.3. Skorokhod problem. Here we present necessary tools to construct the NE strategies under the additional assumptions H1′–H2′. The key to the analysis is the weak construction of a reflected Brownian motion in a general domain, due to Kang and Williams [23]. To proceed further, we need a few vocabularies.

Let $G = \cap_{i\in\mathcal{I}} G_i$ be a nonempty domain in $\mathbb{R}^{n+m}$, where $\mathcal{I}$ is a nonempty finite index set and for each $i \in \mathcal{I}$, G_i is a nonempty domain in $\mathbb{R}^{n+m}$. For simplicity, we assume that $\mathcal{I} = \{1, 2, \ldots, I\}$ with $|\mathcal{I}| = I$. For each $i \in \mathcal{I}$, let $\boldsymbol{n}_i : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ be the unit normal vector field on ∂G_i that points into G_i. And denote $\boldsymbol{r}_i(\cdot) : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ as the reflection direction on ∂G_i. Fix $\boldsymbol{b} \in \mathbb{R}^n$ and $\boldsymbol{\sigma} \in \mathbb{R}^{n\times n}$ as the constant drift and covariance of the diffusion process without reflection. Let ν denote a probability measure on $(\overline{G}, \mathcal{B}(\overline{G}))$, where $\mathcal{B}(\overline{G})$ is the Borel σ-algebra on $\overline{G}$.

A Skorokhod problem is to find a reflected diffusion process in $\overline{G}$ such that the initial distribution follows ν, the diffusion parameters are $(\boldsymbol{b}, \boldsymbol{\sigma})$, and the reflection direction is $\boldsymbol{r}_i$ on face ∂G_i. For each reflection direction $\boldsymbol{r}_i$ $(i \in \mathcal{I})$, denote $\boldsymbol{r}_i^+ := (r_{i,1}, \ldots, r_{i,n})$ as the vector of the first n components of $\boldsymbol{r}_i$ and denote $\boldsymbol{r}_i^- := (r_{i,n+1}, \ldots, r_{i,n+m})$ as the vector of the next m components of $\boldsymbol{r}_i$. Note that $r_{i,k}^- = r_{i,k+n}$ by the usual index rule $(k = 1, \ldots, m)$. Specific to the stochastic game, the following definition is a straightforward modification of [23, Definition 2.1].

Definition 3.2. *A constrained semimartingale reflected Brownian motion (SRBM) associated with the data $(G, \boldsymbol{b}, \boldsymbol{\sigma}, \{\boldsymbol{r}_i\}_{i=1}^I, \nu)$ is an $\{\mathcal{F}_t\}$-adapted, n-dimensional process $\boldsymbol{X}$ defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ such that*

(i) *$\mathbb{P}$-a.s., $\boldsymbol{X}_t = \boldsymbol{W}_t + \sum_{i\in\mathcal{I}}\int_{[0,t)} \boldsymbol{r}_i^+(\boldsymbol{X}_s, \boldsymbol{Y}_s)d\eta_s^i$ for all $t \geq 0$,*

(ii) *under $\mathbb{P}$, $\boldsymbol{W}_t$ is an n-dimensional $\mathcal{F}_t$-Brownian motion with drift vector $\boldsymbol{b}$, covariance matrix $\boldsymbol{\sigma}$, and initial distribution ν,*

(iii) $dY_t^j = \sum_{i\in\mathcal{I}} \int_{[0,t)} \boldsymbol{r}_{i,j}^-(\boldsymbol{X}_t, \boldsymbol{Y}_t) d\eta_t^i$ *and* $Y_t^j \geq 0$ *for* $j = 1, 2, \ldots, m$,

(iv) *for each* $i \in \mathcal{I}$, η^i *is a one-dimensional process such that* $\mathbb{P}$*-a.s.*,

  (a) η^i *is continuous and nondecreasing with* $\eta_0^i = 0$,

  (b) $\eta_t^i = \int_{(0,t]} 1_{\{(\boldsymbol{X}_s,\boldsymbol{Y}_s)\in\partial G_i\cap\partial G\}} d\eta_s^i$ *for all* $t \geq 0$,

(v) $\mathbb{P}$*-a.s.*, $(\boldsymbol{X}_t, \boldsymbol{Y}_t)$ *has continuous paths and* $(\boldsymbol{X}_t, \boldsymbol{Y}_t) \in \overline{G}$ *for all* $t \geq 0$,

Here $\boldsymbol{X}_t$ is the controlled diffusion process and $\boldsymbol{Y}_t$ is the resource levels. The domain G restricts the dynamics of both $\boldsymbol{X}_t$ and $\boldsymbol{Y}_t$.

For each $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{n+m}$, let $\mathcal{I}(\boldsymbol{x}, \boldsymbol{y}) = \{i \in \mathcal{I} : (\boldsymbol{x}, \boldsymbol{y}) \in \partial G_i\}$. Let $U_\epsilon(S)$ denote the closed set $\{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{n+m} : dist((\boldsymbol{x}, \boldsymbol{y}), S) \leq \epsilon\}$ for any $\epsilon > 0$ and $S \subset \mathbb{R}^{n+m}$. If $S = \emptyset$, set $U_\epsilon(S) = \emptyset$ for any $\epsilon > 0$. We list the following assumptions on domain G and reflection directions $\{\boldsymbol{r}_i, i \in \mathcal{I}\}$:

A1. G is the nonempty domain in $\mathbb{R}^{n+m}$ such that $G = \cap_{i\in\mathcal{I}} G_i$, where for each $i \in \mathcal{I}$, G_i is a nonempty domain in $\mathbb{R}^{n+m}$, $G_i \neq \mathbb{R}^{m+n}$, and the boundary ∂G_i is $\mathcal{C}^1$.

A2. For each $\epsilon \in (0, 1)$ there exists $R(\epsilon) > 0$ such that for each $i \in \mathcal{I}$, $(\boldsymbol{x}, \boldsymbol{y}) \in \partial G_i \cap \partial G$, and $(\boldsymbol{x}', \boldsymbol{y}') \in \overline{G}$ satisfying $\|(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x}', \boldsymbol{y}')\| < R(\epsilon)$, we have

$$\langle \boldsymbol{n}_i(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}') - (\boldsymbol{x}, \boldsymbol{y})\rangle \geq -\epsilon\|(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x}', \boldsymbol{y}')\|.$$

A3. The function $D : [0, \infty) \to [0, \infty]$ is such that $D(0) = 0$ and

$$D(\epsilon) = \sup_{\mathcal{I}_0\in\mathcal{I},\mathcal{I}_0\neq\emptyset} \sup\{\text{dist}((\boldsymbol{x}, \boldsymbol{y}), \cap_{i\in\mathcal{I}_0}(\partial G_i \cap \partial G)) : (\boldsymbol{x}, \boldsymbol{y}) \in \cap_{i\in\mathcal{I}_0} U_\epsilon(\partial G_i \cap \partial G)\}$$

for $\epsilon > 0$ satisfies $D(\epsilon) \to 0$ as $\epsilon \to 0$.

A4. There is a constant $L > 0$ such that for each $i \in \mathcal{I}$, $\boldsymbol{r}_i(\cdot)$ is a uniformly Lipschitz continuous function from $\mathbb{R}^{n+m}$ into $\mathbb{R}^{n+m}$ with Lipschitz constant L and $\|\boldsymbol{r}_i(\boldsymbol{x}, \boldsymbol{y})\| = 1$ for each $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{n+m}$.

A5. There is a constant $a \in (0, 1)$, and vector valued function $\boldsymbol{c}(\cdot) = (c_1(\cdot), \ldots, c_I(\cdot))$ and $\boldsymbol{d}(\cdot) = (d_1(\cdot), \ldots, d_I(\cdot))$ from ∂G into $\mathbb{R}_+^I$ such that for each $(\boldsymbol{x}, \boldsymbol{y}) \in \partial G$,

  (i) $\sum_{i\in\mathcal{I}(\boldsymbol{x},\boldsymbol{y})} c_i(\boldsymbol{x}, \boldsymbol{y}) = 1$, $\min_{k\in\mathcal{I}(\boldsymbol{x},\boldsymbol{y})} \langle \sum_{i\in\mathcal{I}(\boldsymbol{x},\boldsymbol{y})} c_i(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{n}_i(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{r}_k(\boldsymbol{x}, \boldsymbol{y})\rangle \geq a$,

  (ii) $\sum_{i\in\mathcal{I}(\boldsymbol{x},\boldsymbol{y})} d_i(\boldsymbol{x}, \boldsymbol{y}) = 1$, $\min_{k\in\mathcal{I}(\boldsymbol{x},\boldsymbol{y})} \langle \sum_{i\in\mathcal{I}(\boldsymbol{x},\boldsymbol{y})} d_i(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{r}_i(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{n}_k(\boldsymbol{x}, \boldsymbol{y})\rangle \geq a$.

THEOREM 3.3. *Given assumptions* A1–A5*, there exists a constrained SRBM associated with the data* $(G, \boldsymbol{b}, \boldsymbol{\sigma}, \{\boldsymbol{r}_i, i \in \mathcal{I}\}, \nu)$.

The proof of Theorem 3.3 is easily adapted from [23, Theorem 5.1], where one constructs a sequence of approximation (random walks) to the constrained SRBM and uses the invariance principle to establish the weak convergence.

4. Nash equilibrium for game $\boldsymbol{C_p}$.

This section analyzes the NE of game $\boldsymbol{C_p}$. Section 4.1 derives the solution to the HJB equations. Section 4.2 constructs the controlled process from the HJB solution. Section 4.3 derives the NE for the game $\boldsymbol{C_p}$. Recall that in game $\boldsymbol{C_p}$, $\boldsymbol{A} = [1, 1, \ldots, 1]^T \in \mathbb{R}^{N\times 1}$, and the unique resource

$$Y_t = y - \sum_{i=1}^N \check{\xi}_t^i \quad \text{and} \quad Y_{0-} = y. \tag{4.1}$$

4.1. Solving HJB equations.

Define

$$\widetilde{x}^i := x^i - \frac{\sum_{j\neq i} x^j}{N-1} \quad \text{for } 1 \leq i \leq N \tag{4.2}$$

to be the relative position from x^i to the center of $(x^j)_{j\neq i}$. For game $\boldsymbol{C_p}$, if $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$, the HJB system simplifies to

$$\text{(HJB-}C_p\text{)} \begin{cases} \min\left\{-\alpha v^i + h\left(\frac{N-1}{N}\widetilde{x}^i\right) + \frac{1}{2}\sum_{j=1}^N v^i_{x^j x^j}, -v^i_y + v^i_{x^i}, -v^i_y - v^i_{x^i}\right\} = 0 \\ \qquad\qquad\qquad\qquad \text{for } (\boldsymbol{x}, y) \in \mathcal{W}_{-i}, \\ -v^i_y - v^i_{x^j} = 0 \qquad \text{for } (\boldsymbol{x}, y) \in \mathcal{A}^+_j, j \neq i, \\ -v^i_y + v^i_{x^j} = 0 \qquad \text{for } (\boldsymbol{x}, y) \in \mathcal{A}^-_j, j \neq i. \end{cases}$$

Now we look for a threshold function $f_N : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\text{(4.3)} \qquad \begin{aligned} & f_N \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}), \quad f'_N(x) < 0 \text{ for } x > 0, \\ & \lim_{x\downarrow 0} f_N(x) = \infty, \text{ and } \exists!\ x_0 > 0 \text{ such that } f_N(x_0) = 0. \end{aligned}$$

Note that for such $f_N(x)$ satisfying condition (4.3), $z - f_N(z) = \tilde{x}^i - y$ has a unique positive root when $\tilde{x}^i \geq f_N^{-1}(y)$, denoted as x^i_+. We consider an even extension of $f_N(x)$ to $(-\infty, 0)$ by defining $\tilde{f}_N(x) = f_N(-x)$ for $x < 0$. Then by symmetry, $z + \tilde{f}_N(z) = \tilde{x}^i + y$ has a unique negative root when $\tilde{x}^i \leq -f_N^{-1}(y)$, denoted as x^i_-. See Figure 2 for an illustration. In particular, we have $f_N(x^i_+) \geq 0$ when $y \geq x_0 + \tilde{x}^i$ and $\tilde{x}^i \geq 0$. Similarly $\tilde{f}_N(x^i_-) \geq 0$ holds when $y \geq -x_0 - \tilde{x}^i$ and $\tilde{x}^i \leq 0$. Such an f_N is constructed later in (4.12) and condition (4.3) is verified in Lemma 4.2.

Then the action region $\mathcal{A}_i$ and the waiting region $\mathcal{W}_i$ of player i are specified as

$$\text{(4.4)} \quad \mathcal{A}^+_i := E^+_i \cap Q_i, \mathcal{A}^-_i := E^-_i \cap Q_i, \mathcal{A}_i = \mathcal{A}^+_i \cup \mathcal{A}^-_i, \mathcal{W}_i := (\mathbb{R}^N \times \mathbb{R}_+) \setminus \mathcal{A}_i$$

with $E^+_i = \{(\boldsymbol{x}, y) \in \mathbb{R}^N \times \mathbb{R}^*_+ : \widetilde{x}^i \geq f_N^{-1}(y)\}$ and $E^-_i = \{(\boldsymbol{x}, y) \in \mathbb{R}^N \times \mathbb{R}^*_+ : \widetilde{x}^i \leq -f_N^{-1}(y)\}$ such that

$$\text{(4.5)} \quad E^+_{i,1} = \left\{(\boldsymbol{x}, y) \in E^+_i : y \geq \tilde{x}^i + x_0\right\}, E^+_{i,2} = \left\{(\boldsymbol{x}, y) \in E^+_i : y < \tilde{x}^i + x_0\right\},$$

$$\text{(4.6)} \quad E^-_{i,1} = \left\{(\boldsymbol{x}, y) \in E^-_i : y \geq -\tilde{x}^i - x_0\right\}, E^-_{i,2} = \left\{(\boldsymbol{x}, y) \in E^+_i : y < -\tilde{x}^i - x_0, \right\},$$

and $\{Q_i\}_{i=1}^N$ disjoint and convex partitions of $\mathbb{R}^N \times \mathbb{R}_+$ such that $Q_i \cap Q_j = (E^+_i \cup E^-_i) \cap (E^+_j \cup E^-_j) \cap \partial\mathcal{W}_{NE}$ for $i \neq j$, $\cup_{i=1}^N Q_i = \mathbb{R}^N \times \mathbb{R}_+$, and $\alpha\boldsymbol{p} + (1-\alpha)\boldsymbol{q} \in Q_j$ for all $\alpha \in [0, 1]$ if $\boldsymbol{p} \in Q_j$ and $\boldsymbol{q} \in Q_j$ for some $j = 1, 2, \ldots, N$. Condition $Q_i \cap Q_j = (E^+_i \cup E^-_i) \cap (E^+_j \cup E^-_j) \cap \partial\mathcal{W}_{NE}$ for $i \neq j$ implies that player i and player j can not jump simultaneously but may apply continuous control (on the boundary of the common waiting region) at the same time. We can define the following mapping:

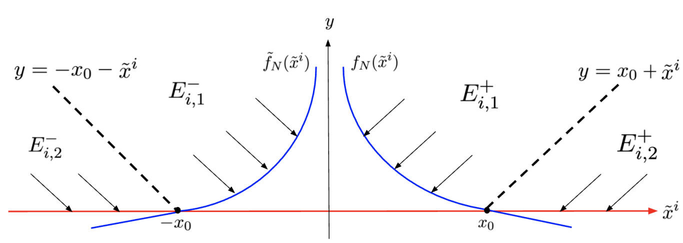


FIG. 2. *Demonstration of the initial control when* $(\boldsymbol{X}_{0-}, Y_{0-}) = (\boldsymbol{x}, y) \notin \overline{\mathcal{W}_{NE}}$.

$$\Pi(\boldsymbol{x},y)=\begin{cases}\left(\left(\boldsymbol{x}^{-i},x_+^i+\frac{\sum_{k\neq i}x^k}{N-1}\right),f_N(x_+^i)\right) & \text{if} \quad (\boldsymbol{x},y)\in Q_i\cap E_{i,1}^+,\\ \left((\boldsymbol{x}^{-i},x^i-y),0\right) & \text{if} \quad (\boldsymbol{x},y)\in Q_i\cap E_{i,2}^+,\\ \left(\left(\boldsymbol{x}^{-i},\frac{\sum_{k\neq i}x^k}{N-1}+x_-^i\right)\tilde{f}_N(x_-^i)\right) & \text{if} \quad (\boldsymbol{x},y)\in Q_i\cap E_{i,1}^-,\\ \left((\boldsymbol{x}^{-i},x^i+y),0\right) & \text{if} \quad (\boldsymbol{x},y)\in Q_i\cap E_{i,2}^-.\end{cases} \tag{4.7}$$

Note that $\Pi(\cdot)$ translates $(\boldsymbol{x},y)$ to the boundary of $E_{i,1}^+$, i.e., $\partial E_{i,1}^+ := \{(\boldsymbol{x},y)\in\mathbb{R}^N\times\mathbb{R}_+ \ : \ y=f_N(\tilde{x}^i), 0<x\le x_0\}$ when $(\boldsymbol{x},y)\in Q_i\cap E_{i,1}^+$, and translates $(\boldsymbol{x},y)$ to the "zero resource" plane $\{(\boldsymbol{x},y)\in\mathbb{R}^N\times\mathbb{R}_+ \ : \ y=0\}$ when $(\boldsymbol{x},y)\in Q_i\cap E_{i,2}^+$, both along the direction $(0,\dots,-1,0,\dots,-1)\in\mathbb{R}^{N+1}$ nonzero ith and $(N+1)$th components. Let

$$\mathcal{W}_{NE} := \cap_{i=1}^N \left(E_i^-\cup E_i^+\right)^c \tag{4.8}$$
$$=\{(\boldsymbol{x},y)\in\mathbb{R}^{N+1}: |\widetilde{x}^i|< f_N^{-1}(y) \text{ with } y>0,\ 1\le i\le N\}\cup\{(\boldsymbol{x},y)\in\mathbb{R}^N\times\mathbb{R}_+ \ : \ y=0\}$$

be the common nonaction region and assume that partitions $\{Q_i\}_{i=1}^N$ satisfies the following assumption:

H3-$\mathbf{C_p}$. For any $(\boldsymbol{x},y)\in\cup_i\mathcal{A}_i$, $\qquad \Pi(\boldsymbol{x},y)\in\overline{\mathcal{W}_{NE}}$.

Condition H3-$\mathbf{C_p}$ implies that if $(\boldsymbol{x},y)\in\mathcal{A}_i$, then the dynamics will be in region $\overline{\mathcal{W}_{NE}}$ after player i's control. For the special case of $N=2$, we can take $Q_1=\{(x_1,x_2,y)\in\mathbb{R}^2\times\mathbb{R}_+ | x_1-x_2\ge 0\}$ and $Q_2=\{(x_1,x_2,y)\in\mathbb{R}^2\times\mathbb{R}_+|x_2-x_1>0\}$. Thus assumption H3-$\mathbf{C_p}$ is easily satisfied. The verification can be found in Appendix B of the online supplementary material.

We seek a solution $v^i(\boldsymbol{x},y)\in\mathcal{C}^2(\overline{\mathcal{W}_{-i}})$ such that if $|\widetilde{x}^i|< f_N^{-1}(y)$, it is of the form

$$v^i(\boldsymbol{x},y)=p_N(\widetilde{x}^i)+A_N(y)\cosh\left(\alpha_N\widetilde{x}^i\right), \text{ where} \tag{4.9}$$

$$p_N(x)=\mathbb{E}\int_0^\infty e^{-\alpha t}h\left(\frac{N-1}{N}x+\sqrt{\frac{N-1}{N}}B_t\right)dt, \alpha_N=\sqrt{\frac{2(N-1)\alpha}{N}} \tag{4.10}$$

with B_t being a one-dimensional Brownian motion. Note that $p_N(\widetilde{x}^i)$ is a solution to $-\alpha v^i + h(\frac{N-1}{N}\widetilde{x}^i)+\frac{1}{2}\sum_{j=1}^N v^i_{x^jx^j}=0$, which corresponds to the waiting region, and $\cosh(\alpha_N\widetilde{x}^i)$ is a solution to $-\alpha v^i+\frac{1}{2}\sum_{j=1}^N v^i_{x^jx^j}=0$. If there is no resource, then $v^i(\boldsymbol{x},y)=p_N(\widetilde{x}^i)$, so $A_N(0)=0$. The following lemma summarizes basic properties of p_N, which can be verified by straightforward calculations.

LEMMA 4.1. *Under assumptions* H1′–H2′, $p_N(x)$ *defined in* (4.10) *satisfies*

$$p_N'(x)\ge 0 \textit{ and } p_N'''(x)\le 0 \textit{ for } x\ge 0 p_N(x)=p_N(-x) \textit{ and } \frac{k}{\alpha}\le p_N''(x)\le\frac{K}{\alpha} \textit{ for } x\in\mathbb{R}.$$

The *smooth-fit principle* states that, along the boundary $y=f_N(\tilde{x}^i)$ between the continuation set $\mathcal{W}_{-i}$ and the action set $\mathcal{A}_i$, v^i has certain regularity properties across the hyperplane. Now applying the smooth-fit principle, we get $v^i_{x^ix^i}=v^i_{yy}=-v^i_{x^iy}$ at the boundary $y=f_N(\widetilde{x}^i)$ with $\widetilde{x}^i>0$. This follows from $v^i_{x^i}+v^i_y=0$ and we expect $v^i\in\mathcal{C}^2(\mathcal{W}_{-i})$. To see this, we differentiate the form (4.9) twice, and the conditions $v^i_{x^i}+v^i_y=0$ and $v^i_{x^ix^i}+v^i_{x^iy}=0$ at the boundary $y=f_N(\widetilde{x}^i)$ lead to

$$\begin{cases} A_N'(f_N(x))=-p_N'(x)\cosh(\alpha_N x)+p_N''(x)\frac{1}{\alpha_N}\sinh(\alpha_N x)\Big|_{x=f_N^{-1}(y)},\\ A_N(f_N(x))=p_N'(x)\frac{1}{\alpha_N}\sinh(\alpha_N x)-p_N''(x)\frac{1}{\alpha_N^2}\cosh(\alpha_N x)\Big|_{x=f_N^{-1}(y)}.\end{cases} \tag{4.11}$$

As a consequence,

$$f_N'(x) = \frac{p_N'(x) - \frac{1}{\alpha_N^2} p_N'''(x)}{p_N''(x)\frac{1}{\alpha_N}\tanh(\alpha_N x) - p_N'(x)} \text{ and} \tag{4.12}$$

$$A_N(y) = p_N'(x)\frac{1}{\alpha_N}\sinh(\alpha_N x) - p_N''(x)\frac{1}{\alpha_N^2}\cosh(\alpha_N x)\Big|_{x=f_N^{-1}(y)}. \tag{4.13}$$

LEMMA 4.2. *Under assumptions* H1′–H2′*,* f_N *defined in* (4.12) *satisfies condition* (4.3)*. Moreover, the curve* $y = f_N(x)$ *intersects* $\{x > 0\}$ *at* x_0 *such that* $A_N(f_N(x_0)) = 0$ *and* x_0 *is the unique positive root of*

$$\alpha_N \tanh(\alpha_N z) = p_N''(z)/p_N'(z). \tag{4.14}$$

Lemma 4.2 can be shown by straightforward calculations and we refer the reader to the online supplementary material for the complete proof.

4.2. Controlled dynamics. Given the candidate game value to (HJB-C_p), we derive the corresponding NEP by showing the existence of a weak solution $(\boldsymbol{X}_t, Y_t)$ to a Skorokhod problem with an unbounded domain, where the boundary of the domain depends on both the diffusion term $\boldsymbol{X}_t$ and the degenerate term $\boldsymbol{Y}_t$.

Recall the region $\mathcal{W}_{NE}$ defined in (4.8) and note that $\mathcal{W}_{NE}$ is unbounded in $\mathbb{R}^{N+1}$ with $2N$ boundaries. For $i = 1, 2, \ldots, N$, define the $2N$ faces of $\mathcal{W}_{NE}$ as

$$F_i = \{(\boldsymbol{x}, y) \in \partial\mathcal{W}_{NE} \mid (\boldsymbol{x}, y) \in \partial E_i^+\}, \quad F_{i+N} = \{(\boldsymbol{x}, y) \in \partial\mathcal{W}_{NE} \mid (\boldsymbol{x}, y) \in \partial E_i^-\}.$$

Then the normal direction of each face is given by $\boldsymbol{n}_i = c_i(-\frac{1}{N-1}, \ldots, -\frac{1}{N-1}, 1, -\frac{1}{N-1}, \ldots, -\frac{1}{N-1}, (f_N^{-1})'(y))$ and $\boldsymbol{n}_{i+N} = c_{i+N}(\frac{1}{N-1}, \ldots, \frac{1}{N-1}, -1, \frac{1}{N-1}, \ldots, \frac{1}{N-1}, (f_N^{-1})'(y))$, with the ith component being ± 1 ($i = 1, 2, \ldots, N$). c_i, c_{N+i} are normalizing constants such that $\|\boldsymbol{n}_i\| = \|\boldsymbol{n}_{N+i}\| = 1$.

Denote the reflection direction on each face as $\boldsymbol{r}_i = c_i'(0, \ldots, -1, \ldots, 0, -1)$ and $\boldsymbol{r}_{N+i} = c_{N+i}'(0, \ldots, 1, \ldots, 0, -1)$ with the ith component being ± 1. c_i', c_{N+i}' are normalizing constants such that $\|\boldsymbol{r}_i\| = \|\boldsymbol{r}_{N+i}\| = 1$. The NE strategy is defined as follows.

Case 1: $(\boldsymbol{X}_{0-}, Y_{0-}) = (\boldsymbol{x}, y) \in \overline{\mathcal{W}_{NE}}$. One can check that $\mathcal{W}_{NE}$ defined in (4.8) and $\{\boldsymbol{r}_i\}_{i=1}^{2N}$ defined above satisfy assumptions A1–A5. (See Appendix A in the online supplementary material for the proof). According to Theorem 3.3, there exists a weak solution to the Skorokhod problem with data $(\mathcal{W}_{NE}, \{\boldsymbol{r}_i\}_{i=1}^{2N}, \boldsymbol{b}, \boldsymbol{\sigma}, \boldsymbol{x} \in \overline{\mathcal{W}_{NE}})$.

Case 2: $(\boldsymbol{X}_{0-}, Y_{0-}) = (\boldsymbol{x}, y) \notin \overline{\mathcal{W}_{NE}}$, that is, there exists $i \in \{1, \ldots, N\}$ such that $(\boldsymbol{X}_{0-}, Y_{0-}) \in \mathcal{A}_i$. (1) If $(\boldsymbol{x}, y) \in \mathcal{A}_i^+ \cap E_{i,1}^+$, then $\tilde{x}^i \geq f_N^{-1}(y)$ and $y \geq \tilde{x}^i + x_0$. In this case, player i will move immediately from $X_{0-}^i = x^i$ to $X_0^i = x_+^i + \frac{\sum_{k\neq i} x^k}{N-1}$ at time 0, where x_+^i is the unique positive root such that $z - f_N(z) = \tilde{x}^i - y$. This will reduce the initial resource from $Y_{0-} = y$ to $Y_0 = f_N(x_+^i) \geq 0$. $f_N(x_+^i) \geq 0$ holds since $y \geq x_0 + \tilde{x}_i$ when $(\boldsymbol{x}, y) \in E_{i,1}^+$. Other players' dynamics remain unchanged, i.e., $X_{0-}^j = X_0^j = x^j$ for $j \neq i$ and $1 \leq j \leq N$. By assumption H3-$\mathbf{C_p}$, we have $(\boldsymbol{X}_0, Y_0) = ((\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} + x_+^i), f_N(x_+^i)) = \Pi(\boldsymbol{X}_{0-}, Y_{0-}) \in \overline{\mathcal{W}_{NE}}$. (2) If $(\boldsymbol{x}, y) \in \mathcal{A}_i^+ \cap E_{i,2}^+$, then $\tilde{x}^i \geq f_N^{-1}(y)$ and $y < \tilde{x}^i + x_0$. In this case, player i will move immediately from $X_{0-}^i = x^i$ to $X_0^i = x^i - y$ and the initial resource $Y_{0-} = y$ is decreased to $Y_0 = 0$ at time 0. Other players' dynamics remain unchanged, i.e.,

$X_0^j = X_{0-}^j = x^j$ for $j \neq i$ and $1 \leq j \leq N$. By assumption H3-$\mathbf{C_p}$, we have $(\boldsymbol{X}_0, Y_0) = ((\boldsymbol{x}^{-i}, x^i - y), 0) = \Pi(\boldsymbol{X}_{0-}, Y_{0-}) \in \overline{\mathcal{W}_{NE}}$. (3) Similarly, if $(\boldsymbol{x}, y) \in \mathcal{A}_i^- \cap E_{i,1}^-$, then $\tilde{x}^i \leq -f_N^{-1}(y)$ and $y \geq -\tilde{x}^i - x_0$. And player i will move immediately from $X_{0-}^i = x^i$ to $X_0^i = x_-^i + \frac{\sum_{k\neq i} x^k}{N-1}$ at time 0, where x_-^i is the unique negative root such that $z + \tilde{f}_N(z) = \tilde{x}^i + y$, and $Y_{0-} = y$ is now $Y_0 = \tilde{f}_N(x_-^i) \geq 0$. Other players' dynamics remain unchanged, i.e., $X_0^j = X_{0-}^j = x^j$ for $j \neq i$ and $1 \leq j \leq N$. By assumption H3-$\mathbf{C_p}$, we have $(\boldsymbol{X}_0, Y_0) = ((\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} + x_-^i), \tilde{f}_N(x_-^i)) = \Pi(\boldsymbol{X}_{0-}, Y_{0-}) \in \overline{\mathcal{W}_{NE}}$. (4) If $(\boldsymbol{x}, y) \in \mathcal{A}_i^- \cap E_{i,2}^-$, then $\tilde{x}^i \leq -f_N^{-1}(y)$ and $y < -\tilde{x}^i - x_0$. In this case, player i will move immediately from $X_{0-}^i = x^i$ to $X_0^i = x^i + y$ and this will change $Y_{0-} = y$ to $Y_0 = 0$ at time 0. Other players' dynamics remain unchanged, i.e., $X_{0-}^j = X_0^j = x^j$ for $j \neq i$ and $1 \leq j \leq N$. By assumption H3-$\mathbf{C_p}$, we have $(\boldsymbol{X}_0, Y_0) = ((\boldsymbol{x}^{-i}, x^i + y), 0) = \Pi(\boldsymbol{X}_{0-}, Y_{0-}) \in \overline{\mathcal{W}_{NE}}$.

4.3. NE for the N-player game. Combining the results in sections 4.1 and 4.2, and based on the verification theorem developed in section 3, we have the following theorem of the NE for the N-player game (2.8) with constraint (4.1).

THEOREM 4.3 (NE for the N-player game $\boldsymbol{C_p}$). *Assume* H1′–H2′ *and* H3-$\boldsymbol{C_p}$. *Define $u^i \in \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ by*

(4.15)
$$u^i(\boldsymbol{x}, y) = \begin{cases} p_N(\widetilde{x}^i) + A_N(y)\cosh\left(\alpha_N \widetilde{x}^i\right) & \text{if } |\tilde{x}^i| \leq f_N^{-1}(y) \text{ and } y = 0, \\ u^i\left(\left(\boldsymbol{x}^{-i}, x_+^i + \frac{\sum_{k\neq i} x^k}{N-1}\right), f_N(x_+^i)\right) & \text{if } (\boldsymbol{x}, y) \in E_{i,1}^+, \\ u^i\left((\boldsymbol{x}^{-i}, x^i - y), 0\right) & \text{if } (\boldsymbol{x}, y) \in E_{i,2}^+, \\ u^i\left(\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} + x_-^i, \tilde{f}_N(x_-^i)\right) & \text{if } (\boldsymbol{x}, y) \in E_{i,1}^-, \\ u^i\left((\boldsymbol{x}^{-i}, x^i + y), 0\right) & \text{if } (\boldsymbol{x}, y) \in E_{i,2}^-, \end{cases}$$

and define $v^i : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ as

(4.16)
$$v^i(\boldsymbol{x}, y) = \begin{cases} u^i(\boldsymbol{x}, y) & \text{if } (\boldsymbol{x}, y) \in \overline{\mathcal{W}_{-i}}, \\ v^i\left(\boldsymbol{x}^{-j}, x_+^j + \frac{\sum_{k\neq j} x^k}{N-1}, f_N(x_+^j)\right) & \text{if } (\boldsymbol{x}, y) \in \mathcal{A}_j^+ \cap E_{j,1}^+ \text{ for } j \neq i, \\ v^i\left(\boldsymbol{x}^{-j}, x^j - y, 0\right) & \text{if } (\boldsymbol{x}, y) \in \mathcal{A}_j^+ \cap E_{j,2}^+ \text{ for } j \neq i, \\ v^i\left(\boldsymbol{x}^{-j}, \frac{\sum_{k\neq j} x^k}{N-1} + x_-^j, \tilde{f}_N(x_-^j)\right) & \text{if } (\boldsymbol{x}, y) \in \mathcal{A}_j^- \cap E_{j,1}^- \text{ for } j \neq i, \\ v^i\left(\boldsymbol{x}^{-j}, x^j + y, 0\right) & \text{if } (\boldsymbol{x}, y) \in \mathcal{A}_j^- \cap E_{j,2}^- \text{ for } j \neq i, \end{cases}$$

where $\mathcal{A}_i$ and $\mathcal{W}_i$ are given in (4.4), *$E_{i,1}^{\pm}$ and $E_{i,2}^{\pm}$ are given in* (4.5)–(4.6) *with $f_N(\cdot)$ defined by* (4.12)–(4.14), *and $\tilde{f}_N(x) = f_N(-x)$ for $x < 0$; $\widetilde{x}^i$ is defined by* (4.2) *and $A_N(\cdot)$ is defined by* (4.13); *x_+^i is the unique positive root of $z - f_N(z) = \widetilde{x}^i - y$ when $\tilde{x}^i \geq f_N^{-1}(y)$, and x_-^i is the unique negative root of $z + \tilde{f}_N(z) = \widetilde{x}^i + y$ when $\tilde{x}^i < -f_N^{-1}(y)$. Then v^i is the game value associated with an NEP $\boldsymbol{\xi}^* = (\xi^{1*}, \ldots, \xi^{N*})$. That is, $v^i(\boldsymbol{x}, y) = J_{C_p}^i(\boldsymbol{x}, y; \boldsymbol{\xi}^*)$. Moreover, the controlled process $(\boldsymbol{X}^*, Y^*)$ under $\boldsymbol{\xi}^*$ is given in section* 4.2.

Proof. First, $u^i(\boldsymbol{x}, y) \in \mathcal{C}^2(\mathbb{R}^N \times \mathbb{R}_+)$ by construction: the $\mathcal{C}^2$ regularity near $y = 0$ follows from (4.13) and the facts that $f_N^{-1}(y) \to x_0$ as $y \to 0$ and $A_N(f_N(x_0)) = 0$. To see that $z - f_N(z) = \widetilde{x}^i - y$ has a unique positive root, it suffices to prove that f_N is decreasing on $\mathbb{R}_+$, which holds by simple calculations. See the online supplementary material for the complete proof. Now let us check conditions (i)–(vii) in Theorem 3.1.

(i) Based on the analysis in section 4.2, when $(\boldsymbol{x}, y) \in \overline{\mathcal{W}_{NE}}$, the NE strategy is a solution to the Skorokhod problem specified in Case 2, which is a continuous process. When $(\boldsymbol{x}, y) \notin \mathcal{W}_{NE}$, the initial push specified in Case 1 satisfies the "no simultaneous jump" condition. Note when the fuel is used up, the dynamics $\boldsymbol{X}_t$ will become uncontrolled and move freely without control.

(ii) Now we check condition (ii) in the verification theorem, i.e., v^i defined in (4.16) satisfying the QVI (3.8). It consists of the following three steps. The idea is to apply the implicit function theorem and the calculation follows the lemma in [7, p.58].

Step (ii)-1 is to verify that v^i defined in (4.16) satisfies

$$-\alpha v^i + h\left(\frac{N-1}{N}\tilde{x}^i\right) + \frac{1}{2}\sum_{j=1}^{N} v^i_{x^j x^j} \geq 0 \text{ for } (\boldsymbol{x}, \mathrm{y}) \in \overline{\mathcal{W}_{-\mathrm{i}}} \tag{4.17}$$

and that the inequality is strict for $(\boldsymbol{x}, y) \in \mathcal{A}_i$ and the equality holds in $\overline{\mathcal{W}_{NE}}$.

Since $p_N(\tilde{x}^i)$ is a solution to $-\alpha v^i + h(\frac{N-1}{N}\tilde{x}^i) + \frac{1}{2}\sum_{j=1}^N v^i_{x^j x^j} = 0$ and $\cosh(\alpha_N \tilde{x}^i)$ is a solution to $-\alpha v^i + \frac{1}{2}\sum_{j=1}^N v^i_{x^j x^j} = 0$, $p_N(\widetilde{x}^i) + A_N(y)\cosh(\widetilde{x}^i \alpha_N)$ satisfies $-\alpha v^i + h(\frac{N-1}{N}\tilde{x}^i) + \frac{1}{2}\sum_{j=1}^N v^i_{x^j x^j} = 0$. Hence, the equality in (4.17) holds for $(\boldsymbol{x}, y) \in \overline{\mathcal{W}_{NE}}$.

Denote $\boldsymbol{p} = (\boldsymbol{w}, z)$ with $\boldsymbol{w} \in \mathbb{R}^N$ and $z \in \mathbb{R}_+$. When $\boldsymbol{p} \in \mathcal{A}_i^+ \cap E_{i,1}^+$, we have $v^i(\boldsymbol{p}) = v^i(\boldsymbol{q})$, where $\boldsymbol{q} := (\boldsymbol{w}^{-i}, w^i_+ + \frac{\sum_{k\neq i} w^k}{N-1}, f_N(w^i_+)) = \Pi(\boldsymbol{p})$ translates $\boldsymbol{p}$ to the boundary of E_i^+, i.e., $\partial E_i^+ := \{(\boldsymbol{x}, y) \mid y = f_N^{-1}(\tilde{x}^i)\}$ along the direction $(0, 0, \ldots, -1, 0, \ldots, -1) \in \mathbb{R}^{N+1}$ with all components zero except the ith and $(N+1)$th components being -1. Note that when $\boldsymbol{p} = (\boldsymbol{w}, z) \in E_{i,1}^+$, we have $z \geq \tilde{w}_i + x_0$ and $f_N(w^i_+) \geq 0$. (See Figure 2.) By the implicit function theorem, $v^i_{x^i x^i}(\boldsymbol{p}) = \frac{v^i_{x^i x^i}(\boldsymbol{q}) + f'_N(w^i_+) v^i_{x^i y}(\boldsymbol{q})}{1 - f'_N(w^i_+)} = v^i_{x^i x^i}(\boldsymbol{q})$, the last equality holds since $v^i_{x^i x^i} = -v^i_{x^i y}$ on $y = f_N(\tilde{x}^i)$. Similarly, we have $v^i_{x^j x^j}(\boldsymbol{p}) = v^i_{x^j x^j}(\boldsymbol{q})$ for $j \neq i$. Therefore, when $\boldsymbol{p} = (\boldsymbol{w}, z) \in \mathcal{A}_i^+ \cap E_{i,1}^+$, we have $-\alpha v^i(\boldsymbol{p}) + h(\frac{N-1}{N}\tilde{p}^i) + \frac{1}{2}\sum_{j=1}^N v^i_{x^j x^j}(\boldsymbol{p}) = (-\alpha v^i(\boldsymbol{q}) + h(\frac{N-1}{N}\tilde{q}^i) + \frac{1}{2}\sum_{j=1}^N v^i_{x^j x^j}(\boldsymbol{q})) + h(\frac{N-1}{N}\tilde{p}^i) - h(\frac{N-1}{N}\tilde{q}^i) > -\alpha v^i(\boldsymbol{q}) + h(\frac{N-1}{N}\tilde{q}^i) + \frac{1}{2}\sum_{j=1}^N v^i_{x^j x^j}(\boldsymbol{q})$ holds in which $\tilde{q}^i = q^i - \frac{\sum_{j=1, j\neq i}^N q^j}{N-1}$ and $\tilde{p}^i = p^i - \frac{\sum_{j=1, j\neq i}^N p^j}{N-1} = \tilde{w}^i$. The last inequality holds since $\tilde{p}^i > \tilde{q}^i > 0$ and h is convex and symmetric to 0. Now for $\boldsymbol{q} \in \partial E_i^+$, we have $-\alpha v^i(\boldsymbol{q}) + h(\frac{N-1}{N}\tilde{q}^i) + \frac{1}{2}\sum_{j=1}^N v^i_{x^j x^j}(\boldsymbol{q}) = 0$. Therefore, $-\alpha v^i + h(\frac{N-1}{N}\tilde{x}^i) + \frac{1}{2}\sum_{j=1}^N v^i_{x^j x^j} \geq 0$ for $\boldsymbol{p} = (\boldsymbol{w}, z) \in \mathcal{W}_i \cap E_{i,1}^+$. When $\boldsymbol{p} = (\boldsymbol{w}, z) \in \mathcal{A}_i^+ \cap E_{i,2}^+$, we have $v^i(\boldsymbol{p}) = v^i(\boldsymbol{q})$, where $\boldsymbol{q} = (\boldsymbol{w}^{-i}, w^i - z, 0) = \Pi(\boldsymbol{p})$ translates $\boldsymbol{p}$ to $\{(\boldsymbol{x}, y) \in \mathbb{R}^N \times \mathbb{R}_+ \mid y = 0\}$. In this case, $v^i(\boldsymbol{p}) = p_N(\widetilde{w}^i - z) + A_N(0)\cosh((\widetilde{w}^i - z)\alpha_N)$ by definition. Hence, $-\alpha v^i(\boldsymbol{p}) + h(\frac{N-1}{N}\tilde{p}^i) + \frac{1}{2}\sum_{j=1}^N v^i_{x^j x^j}(\boldsymbol{p}) = 0$ holds by straightforward calculation. Similar analysis holds for $\boldsymbol{p} := (\boldsymbol{w}, z) \in \mathcal{A}_i^-$.

Step (ii)-2 is to show

$$v^i_{x^i} + v^i_y \leq 0 \text{ and } -v^i_{x^i} + v^i_y \leq 0, \text{ for } (\boldsymbol{x}, y) \in \overline{\mathcal{W}_{-i}}, \tag{4.18}$$

$$v^i_{x^i} + v^i_y = 0, \text{ for } (\boldsymbol{x}, y) \in \mathcal{A}_i^+ \text{ and } -v^i_{x^i} + v^i_y = 0, \text{ for } (\boldsymbol{x}, y) \in \mathcal{A}_i^-. \tag{4.19}$$

Let us first check (4.19). When $\boldsymbol{p} := (\boldsymbol{w}, z) \in \mathcal{A}_i^+ \cap E_{i,1}^+$, denote $\boldsymbol{q} := (\boldsymbol{w}^i, w^i_+ + \frac{\sum_{k\neq i} w^k}{N-1}, f_N(w^i_+)) = \Pi(\boldsymbol{p})$, which translate $\boldsymbol{p}$ to $\partial E_i^+ := \{(\boldsymbol{x}, y) \mid y = f_N(\tilde{x}^i)\}$ along the direction $(0,0,\dots,-1,0,\dots,-1) \in \mathbb{R}^{N+1}$. Then by the definition of (4.16), $v^i(\boldsymbol{p}) = v^i(\boldsymbol{q}) = u^i(\boldsymbol{q})$, $v^i_{x^i}(\boldsymbol{p}) = \frac{1}{1-f'_N(w^i_+)} v^i_{x^i}(\boldsymbol{q}) + \frac{f'_N(w^i_+)}{1-f'_N(w^i_+)} v^i_y(\boldsymbol{q})$, and $v^i_y(\boldsymbol{p}) = -\frac{1}{1-f'_N(w^i_+)} v^i_{x^i}(\boldsymbol{q}) - \frac{f'_N(w^i_+)}{1-f'_N(w^i_+)} v^i_y(\boldsymbol{q})$. Therefore, $v^i_{x^i}(\boldsymbol{p}) + v^i_y(\boldsymbol{p}) = 0$. When $\boldsymbol{p} := (\boldsymbol{w}, z) \in \mathcal{A}^+_i \cap E^+_{i,2}$, we have $v^i(\boldsymbol{p}) = v^i(\boldsymbol{q})$, where $\boldsymbol{q} := (\boldsymbol{w}^{-i}, w^i - z, 0) = \Pi(\boldsymbol{p})$ translates $\boldsymbol{p}$ to $\{(\boldsymbol{x}, y) \in \mathbb{R}^N \times \mathbb{R}_+ \mid y = 0\}$. In this case, $v^i(\boldsymbol{p}) = p_N(\widetilde{w}^i - z) + A_N(0)\cosh((\widetilde{w}^i - z)\alpha_N)$ by definition. Then $v^i_{x^i}(\boldsymbol{p}) + v^i_y(\boldsymbol{p}) = 0$ holds by straightforward calculations. Similarly, $-v^i_{x^i} + v^i_y = 0$ for $(\boldsymbol{x}, y) \in \mathcal{A}^-_i$. As for (4.18), by symmetry it suffices to check the first inequality for $0 \le \tilde{x}^i \le f_N^{-1}(y)$. In this case, by some straightforward calculation,

$$\begin{aligned} v^i_y + v^i_{x^i} &= A'_N(y)\cosh\left(\tilde{x}^i\alpha_N\right) + p'_N(\tilde{x}^i) + A_N(y)\sinh\left(\tilde{x}^i\alpha_N\right)\alpha_N \\ \le p'_N(\tilde{x}^i)&\left(1 - \cosh\left(\left(f_N^{-1}(y) - \tilde{x}^i\right)\alpha_N\right)\right) + p''_N(f_N^{-1}(y))\frac{1}{\alpha_N}\left[\sinh\left(\left(f_N^{-1}(y) - \tilde{x}^i\right)\alpha_N\right)\right. \\ &\left. - \left(\left(f_N^{-1}(y) - \tilde{x}^i\right)\alpha_N\right)\cosh\left(\left(f_N^{-1}(y) - \tilde{x}^i\right)\alpha_N\right)\right] \le 0. \end{aligned}$$

The last inequality holds since $p'_N(\tilde{x}^i) \ge 0$, $|\tilde{x}^i| \le f_N^{-1}(y)$, and $p''_N(f_N^{-1}(y)) > 0$.

Step (ii)-3 is to check

$$-v^i_y - v^i_{x^j} = 0 \text{ for } (\boldsymbol{x}, y) \in \mathcal{A}^+_j, j \ne i, \text{ and } -v^i_y + v^i_{x^j} = 0 \text{ for } (\boldsymbol{x}, y) \in \mathcal{A}^-_j, j \ne i.$$

By symmetry it is sufficient to check the first gradient condition. When $\boldsymbol{p} := (\boldsymbol{w}, z) \in \mathcal{A}^+_j \cap E^+_{j,1}$, denote $\boldsymbol{q} := (\boldsymbol{w}^j, w^j_+ + \frac{\sum_{k\ne j} w^k}{N-1}, f_N(w^j_+)) = \Pi(\boldsymbol{p})$, which translates $\boldsymbol{p}$ to the boundary of E^+_j, i.e., $\partial E^+_j := \{(\boldsymbol{x}, y) \mid y = f_N^{-1}(\tilde{x}^j)\}$ along the direction $(0,0,\dots,-1,0,\dots,-1) \in \mathbb{R}^{N+1}$ with all components zero except the jth and $(N+1)$th components being -1. Then by the definition of (4.16), we have $v^i(\boldsymbol{p}) = v^i(\boldsymbol{q})$, $v^i_{x^j}(\boldsymbol{p}) = \frac{1}{1-f'_N(\tilde{q}^j)} v^i_{x^j}(\boldsymbol{q}) + \frac{f'_N(\tilde{q}^j)}{1-f'_N(\tilde{q}^j)} v^i_y(\boldsymbol{q})$, and $v^i_y(\boldsymbol{p}) = -\frac{1}{1-f'_N(\tilde{q}^j)} v^i_{x^j}(\boldsymbol{q}) - \frac{f'_N(\tilde{q}^j)}{1-f'_N(\tilde{q}^j)} v^i_y(\boldsymbol{q})$, where $\tilde{q}^i = q^i - \frac{\sum_{j=1,j\ne i}^N q^j}{N-1}$. Therefore, $v^i_{x^j}(\boldsymbol{p}) + v^i_y(\boldsymbol{p}) = 0$. When $\boldsymbol{p} := (\boldsymbol{w}, z) \in \mathcal{A}^+_j \cap E^+_{j,2}$, we have $v^i(\boldsymbol{p}) = v^i(\boldsymbol{q})$, where $\boldsymbol{q} := (\boldsymbol{w}^{-j}, w^j - z, 0) = \Pi(\boldsymbol{p})$ translates $\boldsymbol{p}$ to $\{(\boldsymbol{x}, y) \in \mathbb{R}^N \times \mathbb{R}_+ \mid y = 0\}$. In this case, $v^i(\boldsymbol{p}) = p_N(\widetilde{w}^j - z) + A_N(0)\cosh((\widetilde{w}^j - z)\alpha_N)$ holds by definition, and $v^i_{x^j}(\boldsymbol{p}) + v^i_y(\boldsymbol{p}) = 0$ by straightforward calculations.

(iii) By the construction of Cases 1 and 2, when $(\boldsymbol{x}, y) \notin \overline{\mathcal{W}_{-i}}$, there is a push at time 0 to move the joint position to some point $(\hat{\boldsymbol{x}}, \hat{y}) \in \partial\mathcal{W}_{-i}$ such that $\Delta Y_0 \le y$ when $(\boldsymbol{x}, y) \in \overline{\mathcal{W}_{-i}}$, $(\boldsymbol{\xi}^{-i*}, \xi^i)$ forms a solution to the Skorokhod problem in $\cap_{j\ne i}(E^-_j \cup E^+_j)^c$. It is easy to verify that $\cap_{j\ne i}(E^-_j \cup E^+_j)^c \subset \mathcal{W}_{-i}$ and the Skorokhod problem with $\cap_{j\ne i}(E^-_j \cup E^+_j)^c$ has a weak solution. When the fuel is used up, the dynamics $\boldsymbol{X}_t$ will become uncontrolled and move freely. Therefore, condition (iii) is satisfied.

(iv) Solution (4.16) satisfies the smooth-fit principle in section 4.1; therefore, $v^i \in \mathcal{C}^2(\overline{\mathcal{W}_{-i}})$. Let us define a two-dimensional auxiliary function $\widetilde{v}(x, y) = p_N(x) + A_N(y)\cosh(x\alpha_N)$. We first show that $\widetilde{v}(x, y)$ is convex when $|x| \le f_N^{-1}(y)$ and then show that $v^i(\boldsymbol{x}, y)$ defined in (4.16) is convex in $\overline{\mathcal{W}_{-i}}$.

Step (iv)-1 is to show that $\widetilde{v}(x, y)$ is convex when $|x| \le f_N^{-1}(y)$. By straightforward calculation, $\widetilde{v}_{xx}(x, y) = p''_N(x) + \alpha_N^2 A_N(y)\cosh(x\alpha_N)$, $\widetilde{v}_{xy}(x, y) = \alpha_N A'_N(y) \sinh(x\alpha_N)$, and $\widetilde{v}_{yy}(x, y) = A''_N(y)\cosh(x\alpha_N)$. When $0 \le x < f_N^{-1}(y)$, plugging (4.11) into the formula for $\widetilde{v}_{xx}(x, y)$, we have $\widetilde{v}_{xx}(x, y) = p''_N(x) + p'_N(f_N^{-1}(y))\alpha_N \sinh(f_N^{-1}(y)\alpha_N)\cosh(x\alpha_N) - p''_N(f_N^{-1}(y))\cosh(f_N^{-1}(y)\alpha_N)\cosh(x\alpha_N)$. Given Lemma 4.1, $p'_N(x)$ is concave when $x > 0$. Therefore, for $y \ge 0$, $p'_N(f_N^{-1}(y)) \ge p'_N(0) + p''_N(f_N^{-1}(y))(f_N^{-1}(y) - 0) = p''_N(f_N^{-1}(y))f_N^{-1}(y)$. The last equality holds since $h'(0) = 0$

from assumption H2′. Combining the fact that $\sinh(z) \geq 0$ and $\cosh(z) \geq 0$ when $z \geq 0$, we have

$$\begin{aligned}\widetilde{v}_{xx}(x,y) &\geq p_N''(f_N^{-1}(y))f_N^{-1}(y)\alpha_N \sinh\left(f_N^{-1}(y)\alpha_N\right)\cosh\left(x\alpha_N\right)\\ &\quad + p_N''(x) - p_N''(f_N^{-1}(y))\cosh\left(f_N^{-1}(y)\alpha_N\right)\cosh\left(x\alpha_N\right)\\ &\geq p_N''(x) + p_N''(x)\cosh\left(x\alpha_N\right)\left(x\alpha_N\sinh\left(x\alpha_N\right) - \cosh\left(x\alpha_N\right)\right) &(4.20)\\ &= p_N''(x)\left[1 + z\sinh(z)\cosh(z) - \cosh^2(z)\right]\Big|_{z=x\alpha_N} \geq 0. &(4.21)\end{aligned}$$

(4.20) holds since p_N'' is nonincreasing (Lemma 4.1) and $g_1(z) := z\sinh(z) - \cosh(z)$ is nondecreasing when $z \geq 0$. (4.21) holds since $g_2(z) := 1 + z\sinh(z)\cosh(z) - \cosh^2(z)$ is nonnegative when $z \geq 0$. To see this, $g_2(0) = 0$ and $g_2'(z) = \cosh(z)[z\cosh(z) - \sinh(z)] + z\sinh^2(z) \geq 0$, when $z \geq 0$.

On the other hand, denote $g_3(z) := -p_N'(z)\cosh(z\alpha_N) + p_N''(z)\frac{1}{\alpha_N}\sinh(z\alpha_N)$, and then $g_3'(z) = -\alpha_N p_N'(z)\sinh(z\alpha_N) + p_N'''(z)\frac{1}{\alpha_N}\sinh(z\alpha_N)$. From Lemma 4.1, we have $p_N'(z) \geq 0$ and $p_N'''(z) \leq 0$ when $z \geq 0$, and hence $g_3'(z) \leq 0$ when $z \geq 0$. Along with the fact that $f_N'(z) < 0$ when $z > 0$ from Lemma 4.2, we have $A_N''(y) = g_3'(f_N^{-1}(y))\frac{1}{f_N'(f_N^{-1}(y))} \geq 0$. Therefore, $\widetilde{v}_{yy}(x,y) \geq 0$. Finally, we show that $\widetilde{v}_{xx}\widetilde{v}_{yy} - (\widetilde{v}_{xy})^2 \geq 0$ when $0 \leq x \leq f_N^{-1}(y)$. To see this, denote $z = f_N^{-1}(y)$, $\widetilde{v}_{xx}\widetilde{v}_{yy} - (\widetilde{v}_{xy})^2 = \alpha_N^2(-p_N'\cosh(z\alpha_N) + p_N''\frac{1}{\alpha_N}\sinh(z\alpha_N)) \times (p_N'\cosh(x\alpha_N) - p_N'\cosh(z\alpha_N)) \geq 0$. A similar result holds when $-f_N^{-1}(y) \leq x < 0$ by symmetry.

Step (iv)-2 is to show that $v^i(\boldsymbol{x},y)$ defined in (4.16) is convex in $\overline{\mathcal{W}_{-i}}$. We take player one as an example to show $v^1(\boldsymbol{x},y) = \tilde{v}(\tilde{x}_1,y)$ is convex in $\overline{\mathcal{W}_{-1}}$, where $\tilde{x}_1 = x_1 - \frac{\sum_{k=2}^N x_k}{N-1}$. The convexity of other players' value functions can be verified similarly. When $(\boldsymbol{x},y) \in \overline{\mathcal{W}_{-1}}$, we have $|\tilde{x}_1| \leq y$, hence $\widetilde{v}(\tilde{x}_1,y)$ is nonnegative definite. By chain rule, for $2 \leq k \neq j \leq N$, we have $v^1_{x_1x_1}(\boldsymbol{x},y) = \widetilde{v}_{xx}(\tilde{x}_1,y)$, $v_{x_1x_k}(\boldsymbol{x},y) = -\frac{1}{N-1}\widetilde{v}_{xx}(\tilde{x}_1,y)$, $v_{x_1y}(\boldsymbol{x},y) = \widetilde{v}_{xy}(\tilde{x}_1,y)$, $v_{yy}(\boldsymbol{x},y) = \widetilde{v}_{yy}(\tilde{x}_1,y)$, $v^1_{x_kx_j}(\boldsymbol{x},y) = \frac{1}{(N-1)^2}\widetilde{v}_{xx}(\tilde{x}_1,y)$, $v_{x_1x_k}(\boldsymbol{x},y) = -\frac{1}{N-1}\widetilde{v}_{xx}$, and $v_{x_ky}(\boldsymbol{x},y) = -\frac{1}{N-1}\widetilde{v}_{xy}(\tilde{x}_1,y)$.

Denote $H(\boldsymbol{x},y) := \nabla^2 v^1(\boldsymbol{x},y) \in \mathbb{R}^{(N+1)\times(N+1)}$ as the Hessian matrix of v^1 at some point $(\boldsymbol{x},y) \in \overline{\mathcal{W}_{-1}}$. Then for any $\boldsymbol{d} = (b_1,\ldots,b_N,c) \in \mathbb{R}^{N+1}$, we have $\boldsymbol{d}^T H(\boldsymbol{x},y)\boldsymbol{d} = \boldsymbol{e}^T\tilde{H}(\tilde{x}_1,y)\boldsymbol{e} \geq 0$, where $\boldsymbol{e} = (b_1 - \frac{1}{N-1}\sum_{k=2}^N b_k, c)$ and $\tilde{H}(\tilde{x}_1,y) = \nabla^2\tilde{v}(\tilde{x}_1,y)$. The inequality holds since $\tilde{v}(\tilde{x}_1,y)$ is convex when $|\tilde{x}_1| \leq y$. Hence, v^1 is convex in $\overline{\mathcal{W}_{-1}}$.

(v) Denote $\mathcal{W}_{-i}(y) = \{(\boldsymbol{x},z) : (\boldsymbol{x},z) \in \mathcal{W}_{-i}$ and $z \leq y\}$. $(\boldsymbol{X}_t^{-i*}, X_t^i, Y_t) \in \overline{\mathcal{W}_{-i}(y)}$ holds a.s. when $(\xi_t^{-i*},\xi_t^i) \in \mathcal{S}_N(\boldsymbol{x},y)$. This is because $0 \leq Y_t \leq y$ a.s. for all $t \geq 0$ under $(\xi_t^{-i*},\xi_t^i) \in \mathcal{S}_N(\boldsymbol{x},y)$. First, we show that $v^i_{x_j}$ is bounded for $(\boldsymbol{x},z) \in E_{i,1}^+ \cap \overline{\mathcal{W}_{-i}(y)}$, $(\boldsymbol{x},z) \in E_{i,1}^- \cap \overline{\mathcal{W}_{-i}(y)}$, and $(\boldsymbol{x},z) \in B(y) := \overline{\mathcal{W}_{-i}(y)} \cap \{(\boldsymbol{x},z) : |\tilde{x}^i| \leq f_N^{-1}(z)\}$. For $(\boldsymbol{x},z) \in B(y)$, $|\tilde{x}^i| \leq f_N^{-1}(z) \leq f_N^{-1}(y) < \infty$ since f_N^{-1} is nonincreasing. This implies that $\tilde{x}^i$ is bounded in $B(y)$. By the definition of $A_N(z)$ in (4.9), $A_N(z)$ is bounded in $B(y)$. Hence, $v^i_{x_k}$ is bounded on $B(y)$ $(k = 1,2,\ldots,N)$. Following Step (ii)-2, there exists $\boldsymbol{q} \in \partial B(y)$ such that $v_{x^k}(\boldsymbol{q}) = v_{x^k}(\boldsymbol{x},z)$ $(k = 1,2,\ldots,N)$ for $(\boldsymbol{x},z) \in E_{i,1}^+ \cap \overline{\mathcal{W}_{-i}(y)}$. A similar result holds for $(\boldsymbol{x},z) \in E_{i,1}^- \cap \overline{\mathcal{W}_{-i}(y)}$. Hence, $v^i_{x_k}$ is bounded on $(\boldsymbol{x},z) \in E_{i,1}^+ \cap \overline{\mathcal{W}_{-i}(y)}$ and $(\boldsymbol{x},z) \in E_{i,1}^- \cap \overline{\mathcal{W}_{-i}(y)}$. Second, $v^i(\boldsymbol{x},0) = p_N(\tilde{x}^i)$ holds since $A_N(0) = 0$ (Lemma 4.2). By the definition of v^i and following Step (ii)-2, we have $v^i_{x^k}(\boldsymbol{x},z) = v^i_{x^k}((\boldsymbol{x}^{-i}, x^i - z), 0)$ $(k = 1,2,\ldots,N)$ and $0 < \tilde{x}^i - z < \tilde{x}^i$ for $(\boldsymbol{x},z) \in E_{i,2}^+ \cap \overline{\mathcal{W}_{-i}(y)}$. From Lemma 4.1, $0 \leq p_N'(\tilde{x}^i - z) \leq p_N'(\tilde{x}^i)$. Hence, $|v^i_{x^k}(\boldsymbol{x},z)| \leq |p_N'(\tilde{x}^i)|$ for $(\boldsymbol{x},z) \in E_{i,1}^+ \cap \overline{\mathcal{W}_{-i}(y)}$ and the same result holds for $(\boldsymbol{x},z) \in E_{i,2}^- \cap \overline{\mathcal{W}_{-i}(y)}$. Combining the above analysis with Lemma 4.1, there exists a constant $C(y) > 0$ such that $|v^i_{x^j}(\boldsymbol{x},z)| \leq C(y) + |p_N'(\tilde{x}^i)| \leq C(y) + \frac{K}{\alpha}|\tilde{x}^i|$ for $(\boldsymbol{x},z) \in \overline{\mathcal{W}_{-i}(y)}$.

Hence, by Tonelli's theorem, $\mathbb{E}[\int_0^T e^{-2\alpha t}(v^i_{x^j}(\boldsymbol{X}_t^{-i*}, X_t^i, Y_t))^2 dt] \leq C_0(C^2(y) + (x^i - \frac{\sum_{j\neq i} x_j}{N-1})^2 + y^2 + T) < \infty$ for some $C_0 > 0$ and (v) is satisfied.

(vi) Recall the definition of $\mathcal{W}_{-i}(y)$ in (v) and the fact that $(\boldsymbol{X}_t^{-i*}, X_t^i, Y_t) \in \overline{\mathcal{W}_{-i}(y)}$ when $(\xi_t^{-i*}, \xi_t^i) \in \mathcal{S}_N(\boldsymbol{x}, y)$. Following the same argument as in (v), there exists $\widetilde{C}(y) > 0$ such that $|v^i(\boldsymbol{x}, z)| \leq \widetilde{C}(y)$ for $(\boldsymbol{x}, z) \in E^+_{i,1} \cap \overline{\mathcal{W}_{-i}(y)}$, $(\boldsymbol{x}, z) \in E^-_{i,1} \cap \overline{\mathcal{W}_{-i}(y)}$, and $(\boldsymbol{x}, z) \in B(y) := \overline{\mathcal{W}_{-i}(y)} \cap \{(\boldsymbol{x}, z) : |\tilde{x}^i| \leq f_N^{-1}(z)\}$. In addition, $v^i(\boldsymbol{x}, 0) = p_N(\tilde{x}^i)$ holds since $A_N(0) = 0$ (Lemma 4.2). By the definition of v^i, $v^i(\boldsymbol{x}, z) = v^i((\boldsymbol{x}^{-i}, x^i - z), 0)$ and $0 < \tilde{x}^i - z < \tilde{x}^i$ for $(\boldsymbol{x}, z) \in E^+_{i,2} \cap \overline{\mathcal{W}_{-i}(y)}$. From Lemma 4.1, $0 \leq p_N(\tilde{x}^i - z) \leq p_N(\tilde{x}^i)$. Hence, $v^i(\boldsymbol{x}, z) \leq p_N(\tilde{x}^i)$ for $(\boldsymbol{x}, z) \in E^+_{i,2} \cap \overline{\mathcal{W}_{-i}(y)}$ and the same result holds for $(\boldsymbol{x}, z) \in E^-_{i,2} \cap \overline{\mathcal{W}_{-i}(y)}$. Combining the above analysis with Lemma 4.1, $|v(\boldsymbol{x}, y)| \leq p_N(\tilde{x}^i) + \widetilde{C}(y) \leq p_N(0) + \frac{K}{\alpha}(\tilde{x}^i)^2 + \widetilde{C}(y)$. Given $(\boldsymbol{\xi}^{-i*}, \xi^i) \in \mathcal{S}_N(\boldsymbol{x}, \boldsymbol{y})$, $\sum_{j\neq i} \check{\xi}_T^{j*} + \check{\xi}_T^i \leq y$ holds a.s. Therefore, $\mathbb{E}[(X_T^i - \frac{\sum_{j\neq i} X_T^{j*}}{N-1})^2] \leq \tilde{C}_0((x_0^i - \frac{\sum_{j\neq i} x_j}{N-1})^2 + y^2 + T)$ for some $\tilde{C}_0 > 0$. Hence, $\lim\sup_{T\to\infty} e^{-\alpha T}\mathbb{E}[p_N(X_T^i - \frac{\sum_{j\neq i} X_T^{j*}}{N-1})] = 0$ and condition (vi) holds.

(vii) This condition is satisfied by the property of the Skorokhod problem and the initial jump described in section 4.2. ∎

5. Nash equilibrium for game $\boldsymbol{C_d}$.

In this section, we study the NEP of the N-player game $\boldsymbol{C_d}$. That is, $A = \boldsymbol{I_N} \in \mathbb{R}^{N\times N}$, and

$$Y_t^i = y^i - \check{\xi}_t^i \quad \text{with} \quad Y_{0-}^i = y^i. \tag{5.1}$$

Recall that the major difference between game $\boldsymbol{C_p}$ and game $\boldsymbol{C_d}$ is that in the former all N players share a fixed amount of the same resource, while in the latter each player has her own individual fixed resource constraint. This difference is reflected in $(HJB - C_p)$ and $(HJB - C_d)$ in terms of their dimensionality and in each player's control based on the remaining resources. In particular, $(HJB - C_p)$ and the state space $(\boldsymbol{x}, y)$ of $\boldsymbol{C_p}$ are of dimension $N + 1$, whereas $(HJB - C_d)$ and the state space $(\boldsymbol{x}, \boldsymbol{y})$ of $\boldsymbol{C_d}$ are of dimension $2N$. Moreover, in game $\boldsymbol{C_p}$, the gradient constraint is $-v_y^i \pm v_{x^i}^i$ for player i. In contrast, in game $\boldsymbol{C_d}$, each player controls her own resource level, and the gradient constraint becomes $-v_{y^i}^i \pm v_{x^i}^i$ for player i. So if $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$, the HJB equation for $v^i(\boldsymbol{x}, \boldsymbol{y})$ in game $\boldsymbol{C}_d$ is as follows:

$$\text{(HJB-}C_d\text{)} \begin{cases} \min\left\{-\alpha v^i + h\left(\frac{N-1}{N}\widetilde{x}^i\right) + \frac{1}{2}\sum_{j=1}^N v^i_{x^j x^j}, -v^i_{y^i} + v^i_{x^i}, -v^i_{y^i} - v^i_{x^i}\right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{W}_{-i}, \\ -v^i_{y^j} - v^i_{x^j} = 0 \ \text{ for } (\boldsymbol{x}, y) \in \mathcal{A}_j^+, j \neq i, \text{ and } -v^i_{y^j} + v^i_{x^j} = 0 \ \text{ for } (\boldsymbol{x}, y) \in \mathcal{A}_j^-, j \neq i. \end{cases}$$

Note that the control policy of the ith player only depends on $(\boldsymbol{x}, y^i)$ in $\mathcal{W}_{-i}$. As seen in section 4, for the controlled process of type $\boldsymbol{C_p}$, upon hitting the boundary of the polyhedron, the polyhedron will expand in all directions. While for the controlled process of type $\boldsymbol{C_d}$, only one direction of the the polyhedron will move once hit.

To proceed, similarly to section 4, define the action region $\mathcal{A}_i \in \mathbb{R}^N \times \mathbb{R}_+^N$ and the waiting region $\mathcal{W}_i$ of the ith player by

$$\mathcal{A}_i^+ = E_i^+ \cap Q_i, \quad \mathcal{A}_i^- = E_i^- \cap Q_i \quad, \mathcal{A}_i = \mathcal{A}_i^+ \cup \mathcal{A}_i^-, \text{ and } \quad \mathcal{W}_i = \mathbb{R}^N \times \mathbb{R}_+^N \setminus \mathcal{A}_i \tag{5.2}$$

with $E_i^+ = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^N \times (\mathbb{R}_+^*)^N : \widetilde{x}^i \geq f_N^{-1}(y^i)\}$ and $E_i^- = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^N \times (\mathbb{R}_+^*)^N : \widetilde{x}^i \leq -f_N^{-1}(y^i)\}$,

$$E_{i,1}^{+} = \left\{(\boldsymbol{x},\boldsymbol{y}) \in E_i^{+} : y^i \geq \tilde{x}^i + x_0\right\}, E_{i,2}^{+} = \left\{(\boldsymbol{x},\boldsymbol{y}) \in E_i^{+} : y^i < \tilde{x}^i + x_0\right\}, \tag{5.3}$$

$$E_{i,1}^{-} = \left\{(\boldsymbol{x},\boldsymbol{y}) \in E_i^{-} : y^i \geq -\tilde{x}^i - x_0\right\}, E_{i,2}^{-} = \left\{(\boldsymbol{x},\boldsymbol{y}) \in E_i^{+} : y^i < -\tilde{x}^i - x_0\right\}, \tag{5.4}$$

and $\{Q_i\}_{i=1}^N$ convex partitions of $\mathbb{R}^N \times \mathbb{R}_+$ such that $Q_i \cap Q_j = (E_i^+ \cup E_i^-) \cap (E_j^+ \cup E_j^-) \cap \partial\mathcal{W}_{NE}$ for $i \neq j$, $\cup_{i=1}^N Q_i = \mathbb{R}^N \times \mathbb{R}_+$, and $\alpha\boldsymbol{p} + (1-\alpha)\boldsymbol{q} \in Q_j$ for all $\alpha \in [0,1]$ if $\boldsymbol{p} \in Q_j$ and $\boldsymbol{q} \in Q_j$ for some $j = 1, 2, \ldots, N$. We can define the following mapping,

$$\Pi(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \left(\left(\boldsymbol{x}^{-i}, x_+^i + \frac{\sum_{k\neq i} x^k}{N-1}\right), (\boldsymbol{y}^{-i}, f_N(x_+^i))\right) & \text{if} \quad (\boldsymbol{x},\boldsymbol{y}) \in Q_i \cap E_{i,1}^{+}, \\ \left((\boldsymbol{x}^{-i}, x^i - y^i), (\boldsymbol{y}^{-i}, 0)\right) & \text{if} \quad (\boldsymbol{x},\boldsymbol{y}) \in Q_i \cap E_{i,2}^{+}, \\ \left(\left(\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} + x_-^i\right), (\boldsymbol{y}^{-i}, \tilde{f}_N(x_-^i))\right) & \text{if} \quad (\boldsymbol{x},\boldsymbol{y}) \in Q_i \cap E_{i,1}^{-}, \\ \left((\boldsymbol{x}^{-i}, x^i + y^i), (\boldsymbol{y}^{-i}, 0)\right) & \text{if} \quad (\boldsymbol{x},\boldsymbol{y}) \in Q_i \cap E_{i,2}^{-}, \end{cases} \tag{5.5}$$

in which the threshold function $f_N(\cdot)$ is defined in (4.12)–(4.14), x_+^i is the unique positive root such that $z - f_N(z) = \tilde{x}^i - y^i$, and x_-^i is the unique negative root such that $z + \tilde{f}_N(z) = \tilde{x}^i + y^i$. Note that $\Pi(\cdot)$ translates $(\boldsymbol{x},\boldsymbol{y})$ to the boundary of $E_{i,1}^{+}$, i.e., $\partial E_{i,1}^{+} := \{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}_+^N \ : \ y^i = f_N^{-1}\left(\tilde{x}^i\right), 0 < \tilde{x}^i \leq x_0\}$ when $(\boldsymbol{x},\boldsymbol{y}) \in Q_i \cap E_{i,1}^{+}$, and translates $(\boldsymbol{x},\boldsymbol{y})$ to the zero-resource plane $\{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}_+^N \ : \ y^i = 0\}$ when $(\boldsymbol{x},\boldsymbol{y}) \in Q_i \cap E_{i,2}^{+}$, both along the direction $(0,\ldots,-1,0,\ldots,-1,\ldots,0) \in \mathbb{R}^{2N}$ with nonzero ith and $(N+i)$th components. Let

$$\mathcal{W}_{NE} := \{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}_+^N : |\widetilde{x}^i| < f_N^{-1}(y^i) \text{ for } 1 \leq i \leq N\} \cup \{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}_+^N : \boldsymbol{y} = 0\}, \tag{5.6}$$

and assume $\{Q_i\}_{i=1}^N$ satisfies the following assumption:

H3-$\mathbf{C_d}$. For any $(\boldsymbol{x},\boldsymbol{y}) \in \cup_i \mathcal{A}_i$, $\quad \Pi(\boldsymbol{x},\boldsymbol{y}) \in \overline{\mathcal{W}_{NE}}$.

Condition H3-$\mathbf{C_d}$ implies that if $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_i$, then the dynamics will be in region $\overline{\mathcal{W}_{NE}}$ after player i's control.

We now investigate control of player i, which only depends on $(\boldsymbol{x}, y^i)$ in $\mathcal{W}_{-i}$. That is, for $|\widetilde{x}^i| < f_N^{-1}(y^i)$,

$$v^i(\boldsymbol{x},\boldsymbol{y}) = p_N(\widetilde{x}^i) + A_N(y^i)\cosh\left(\widetilde{x}^i \alpha_N\right) \tag{5.7}$$

is a solution to (HJB-C_d) with $p_N(\cdot)$ defined in (4.10) and $A_N(\cdot)$ defined in (4.13).

The next step is to construct the controlled process $(\boldsymbol{X},\boldsymbol{Y})$ corresponding to (5.7). Note that $\mathcal{W}_{NE}$ is an unbounded domain in $\mathbb{R}^{2N}$ with $2N$ boundaries. For $i = 1, 2, \ldots, N$, define the $2N$ faces of $\mathcal{W}_{NE}$ as $F_i = \{(\boldsymbol{x},\boldsymbol{y}) \in \partial\mathcal{W}_{NE} \mid (\boldsymbol{x},\boldsymbol{y}) \in \partial E_i^+\}$ and $F_{i+N} = \{(\boldsymbol{x},\boldsymbol{y}) \in \partial\mathcal{W}_{NE} \mid (\boldsymbol{x},\boldsymbol{y}) \in \partial E_i^-\}$. The normal direction on each face is given by $\boldsymbol{n}_i = c_i(\frac{1}{N-1},\ldots,\frac{1}{N-1}-1,\frac{1}{N-1}\ldots,\frac{1}{N-1};0,\ldots,0,(f_N^{-1})'(y^i),0,\ldots,0)$, and $\boldsymbol{n}_{N+i} = c_{N+i}(-\frac{1}{N-1},\ldots,-\frac{1}{N-1},1,-\frac{1}{N-1},\ldots,-\frac{1}{N-1};0,\ldots,0,(f_N^{-1})'(y^i),0,\ldots,0)$ with the ith component being ± 1 and the $(N+i)$th component being $(f_N^{-1})'(y^i)$. c_i and c_{N+i} are normalizing constants such that $\|\boldsymbol{n}_i\| = \|\boldsymbol{n}_{N+i}\| = 1$. Denote the reflection direction on each face as $\boldsymbol{r}_i = c_i'(0,\ldots,0,-1,0,\ldots 0;0,\ldots,0,-1,0,\ldots,0)$ and $\boldsymbol{r}_{N+i} = c_{N+i}'(0,\ldots,0,1,0,\ldots 0;0,\ldots,0,-1,0,\ldots,0)$, with the ith component being ± 1 and the $(N+i)^{th}$ component being ± 1. c_i' and c_{N+i}' are normalizing constants such that $\|\boldsymbol{r}_i\| = \|\boldsymbol{r}_{N+i}\| = 1$. The NE strategy is defined as follows.

Case 1: $(\boldsymbol{X}_{0-},\boldsymbol{Y}_{0-}) = (\boldsymbol{x},\boldsymbol{y}) \in \overline{\mathcal{W}_{NE}}$. One can check that $\mathcal{W}_{NE}$ defined in (5.6) and $\{\boldsymbol{r}_i\}_{i=1}^{2N}$ defined above satisfies assumptions A1–A5 (see Appendix A in the online supplementary material for the proof). Therefore, there exists a weak solution to the Skorokhod problem with data $(\mathcal{W}_{NE}, \{\boldsymbol{r}_i\}_{i=1}^{2N}, \boldsymbol{b}, \boldsymbol{\sigma}, \boldsymbol{x} \in \overline{\mathcal{W}_{NE}})$.

Case 2: $(\boldsymbol{X}_{0-}, \boldsymbol{Y}_{0-}) = (\boldsymbol{x}, \boldsymbol{y}) \notin \overline{\mathcal{W}_{NE}}$. There exists $i \in \{1, \dots, N\}$ such that $(\boldsymbol{X}_{0-}, \boldsymbol{Y}_{0-}) \in \mathcal{A}_i$. (1) If $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}_i^+ \cap E_{i,1}^+$, then player i will move immediately from $X_{0-}^i = x^i$ to $X_0^i = x_+^i + \frac{\sum_{k\neq i} x^k}{N-1}$ at time 0, where x_+^i is the unique positive root such that $z - f_N(z) = \tilde{x}^i - y^i$. This will reduce player i's resource from $Y_{0-}^i = y^i$ to $Y_0^i = f_N(x_+^i) \geq 0$. Other players' dynamics and resources remain unchanged, i.e., $X_0^j = X_{0-}^j = x^j$ and $Y_0^j = Y_{0-}^j = y^j$ for $j \neq i$ and $1 \leq j \leq N$. By assumption H3-$\mathbf{C_d}$, we have $(\boldsymbol{X}_0, \boldsymbol{Y}_0) = ((\boldsymbol{x}^{-i}, x_+^i + \frac{\sum_{k\neq i} x^k}{N-1}), (\boldsymbol{y}^{-i}, f_N(x_+^i))) = \Pi((\boldsymbol{X}_{0-}, \boldsymbol{Y}_{0-})) \in \overline{\mathcal{W}_{NE}}$. (2) If $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}_i^+ \cap E_{i,2}^+$, then player i will move immediately from $X_{0-}^i = x^i$ to $X_0^i = x^i - y^i$ and her resource changes from $Y_{0-}^i = y^i$ to $Y_0^i = 0$ at time 0. Other players' positions and resources remain unchanged, i.e., $X_0^j = X_{0-}^j = x^j$ and $Y_0^j = Y_{0-}^j = y^j$ for $j \neq i$ and $1 \leq j \leq N$. By assumption H3-$\mathbf{C_d}$, we have $(\boldsymbol{X}_0, \boldsymbol{Y}_0) = ((\boldsymbol{x}^{-i}, x^i - y^i, (\boldsymbol{y}^{-i}, 0)) = \Pi((\boldsymbol{X}_{0-}, \boldsymbol{Y}_{0-})) \in \overline{\mathcal{W}_{NE}}$. (3) Similarly, if $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}_i^- \cap E_{i,1}^-$, then player i will move immediately from $X_{0-}^i = x^i$ to $X_0^i = x_-^i + \frac{\sum_{k\neq i} x^k}{N-1}$ at time 0, where x_-^i is the unique negative root such that $z + \tilde{f}_N(z) = \tilde{x}^i + y^i$. This will reduce her resource from $Y_{0-}^i = y$ to $Y_0^i = f_N(x_-^i) \geq 0$. Other players' dynamics and resources remain unchanged, i.e., $X_0^j = X_{0-}^j = x^j$ and $Y_0^j = Y_{0-}^j = y^j$ for $j \neq i$ and $1 \leq j \leq N$. By Assumption H3-$\mathbf{C_d}$, we have $(\boldsymbol{X}_0, \boldsymbol{Y}_0) = ((\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} + x_-^i), (\boldsymbol{y}^{-i}, \tilde{f}_N(x_-^i))) = \Pi((\boldsymbol{X}_{0-}, \boldsymbol{Y}_{0-})) \in \overline{\mathcal{W}_{NE}}$. (4) If $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}_i^- \cap E_{i,2}^-$; then player i will move immediately from $X_{0-}^i = x^i$ to $X_0^i = x^i + y^i$ and her resource reduces from $Y_{0-}^i = y^i$ to $Y_0^i = 0$ at time 0. Other players' dynamics and resources remain unchanged. By assumption H3-$\mathbf{C_d}$, we have $(\boldsymbol{X}_0, \boldsymbol{Y}_0) = ((\boldsymbol{x}^{-i}, x^i + y^i), (\boldsymbol{y}^{-i}, 0)) = \Pi((\boldsymbol{X}_{0-}, \boldsymbol{Y}_{0-})) \in \overline{\mathcal{W}_{NE}}$.

In summary, the NE for game (2.8) with constraint $\boldsymbol{C_d}$ is stated as follows.

THEOREM 5.1 (NE for the N-player game $\boldsymbol{C_d}$). *Assume* H1′–H2′ *and* H3–$\boldsymbol{C_d}$. *Define* $u^i \in \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ *as*

(5.8)
$$u^i(\boldsymbol{x}, y) = \begin{cases} p_N(\tilde{x}^i) + A_N(y)\cosh\left(\alpha_N \tilde{x}^i\right) & \text{if } |\tilde{x}^i| \leq f_N^{-1}(y) \text{ and } y = 0, \\ u^i\left(\boldsymbol{x}^{-i}, x_+^i + \frac{\sum_{k\neq i} x^k}{N-1}, f_N(x_+^i)\right) & \text{if } \tilde{x}^i > f_N^{-1}(y) \text{ and } y \geq \tilde{x}^i + x_0, \\ u^i\left(\boldsymbol{x}^{-i}, x^i - y, 0\right) & \text{if } \tilde{x}^i > f_N^{-1}(y) \text{ and } y < \tilde{x}^i + x_0, \\ u^i\left(\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} + x_-^i, \tilde{f}_N(x_-^i)\right) & \text{if } \tilde{x}^i < -f_N^{-1}(y) \text{ and } y \geq -\tilde{x}^i + x_0, \\ u^i\left(\boldsymbol{x}^{-i}, x^i + y, 0\right) & \text{if } \tilde{x}^i < -f_N^{-1}(y) \text{ and } y < -\tilde{x}^i + x_0, \end{cases}$$

and define $v^i : \mathbb{R}^N \times \mathbb{R}_+^N \to \mathbb{R}$ *as*

(5.9)
$$v^i(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} u^i(\boldsymbol{x}, y^i) & \text{if } (\boldsymbol{x}, \boldsymbol{y}) \in \overline{\mathcal{W}_{-i}}, \\ v^i\left(\boldsymbol{x}^{-j}, x_+^j + \frac{\sum_{k\neq j} x^k}{N-1}, (\boldsymbol{y}^{-j}, f_N(x_+^j))\right) & \text{if } (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}_j^+ \cap E_{j,1}^+ \text{ for } j \neq i, \\ v^i\left(\boldsymbol{x}^{-j}, x^j - y^j, (\boldsymbol{y}^{-j}, 0)\right) & \text{if } (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}_j^+ \cap E_{j,2}^+ \text{ for } j \neq i, \\ v^i\left(\boldsymbol{x}^{-j}, \frac{\sum_{k\neq j} x^k}{N-1} + x_-^j, (\boldsymbol{y}^{-j}, \tilde{f}_N(x_-^j))\right) & \text{if } (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}_j^- \cap E_{j,1}^- \text{ for } j \neq i, \\ v^i\left(\boldsymbol{x}^{-j}, x^j + y^j, (\boldsymbol{y}^{-j}, 0)\right) & \text{if } (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}_j^- \cap E_{j,2}^- \text{ for } j \neq i, \end{cases}$$

where $\mathcal{A}_i$ and $\mathcal{W}_i$ are given in (5.2), $E^{\pm}_{i,1}$ *and* $E^{\pm}_{i,2}$ *are given in* (5.3)–(5.4) *with* $f_N(\cdot)$ *defined by* (4.12)–(4.14), *and* $\tilde{f}_N(x) = f_N(-x)$ *for* $x < 0$*;* $\widetilde{x}^i$ *is defined by* (4.2), *and* $A_N(\cdot)$ *is defined by* (4.13)*;* x^i_+ *in* (5.8) *is the unique positive root of* $z - f_N(z) = \widetilde{x}^i - y$ *when* $\tilde{x}^i \geq f_N^{-1}(y)$, *and* x^i_- *is the unique negative root of* $z + \tilde{f}_N(z) = \widetilde{x}^i + y$ *when* $\tilde{x}^i < -f_N^{-1}(y)$*;* x^j_+ *in* (5.9) *is the unique positive root of* $z - f_N(z) = \widetilde{x}^j - y^j$ *if* $\tilde{x}^j \geq f_N^{-1}(y^j)$, *and* x^j_- *is the unique negative root of* $z + \tilde{f}_N(z) = \widetilde{x}^i + y^j$ *if* $\tilde{x}^j < -f_N^{-1}(y^j)$. *Then* v^i *is the game value associated with an NEP* $\boldsymbol{\xi}^* = (\xi^{1*}, \ldots, \xi^{N*})$. *That is,* $v^i(\boldsymbol{x}, \boldsymbol{y}) = J^i_{C_d}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}^*)$. *Moreover, the controlled process* $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ *under* $\boldsymbol{\xi}^*$ *is given in this section:* Case 1 *if* $(\boldsymbol{x}, \boldsymbol{y}) \in \overline{\mathcal{W}_{NE}}$, *and* Case 2 *if* $(\boldsymbol{x}, \boldsymbol{y}) \notin \overline{\mathcal{W}_{NE}}$.

The proof of Theorem 5.1 is similar to that of Theorem 4.3 and hence omitted.

6. Nash equilibrium for game $\boldsymbol{C}$.

In the previous two sections, we have dealt with two special games $\boldsymbol{C_p}$ and $\boldsymbol{C_d}$. Analysis of these two games provides important insight into the solution structure of the general game $\boldsymbol{C}$. Namely, the NE strategy depends on the positions of players and their remaining resource levels. With these two special cases in mind, now recall that in game $\boldsymbol{C}$,

$$dY^j_t = -\sum_{i=1}^{N} \frac{a_{ij} Y^j_{t-}}{\sum_{k=1}^{M} a_{ik} Y^k_{t-}} d\check{\xi}^i_t \quad \text{and} \quad Y^j_{0-} = y^j \geq 0. \tag{6.1}$$

For the HJB equation $(HJB - C)$, the gradient constraint is more complicated than the two special cases $\boldsymbol{C_p}$ and $\boldsymbol{C_d}$. When $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$,

$$\textit{(HJB-C)} \begin{cases} \min\left\{ -\alpha v^i + h + \dfrac{1}{2}\displaystyle\sum_{j=1}^{N} v^i_{x^j x^j}, \Gamma_i v^i + v^i_{x^i}, -\Gamma_i v^i - v^i_{x^i} \right\} = 0 & \text{for } (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{W}_{-i}, \\ -\Gamma_j v^i - v^i_{x^j} = 0 & \text{for } (\boldsymbol{x}, y) \in \mathcal{A}^+_j, j \neq i, \\ -\Gamma_j v^i + v^i_{x^j} = 0 & \text{for } (\boldsymbol{x}, y) \in \mathcal{A}^-_j, j \neq i. \end{cases}$$

In particular, if $\boldsymbol{A} = [1, 1, \ldots, 1]^T \in \mathbb{R}^{N \times 1}$, then $(HJB - C)$ becomes $(HJB - C_p)$; and if $\boldsymbol{A} = \boldsymbol{I_N}$, then it is $(HJB - C_d)$.

Similarly to section 4, define the action and the waiting regions $\mathcal{A}_i$ and $\mathcal{W}_i$ of player i by

$$\mathcal{A}^+_i := E^+_i \cap Q_i, \quad \mathcal{A}^-_i := E^-_i \cap Q_i \quad, \mathcal{A}_i = \mathcal{A}^+_i \cup \mathcal{A}^-_i, \quad \text{and} \quad \mathcal{W}_i := \mathbb{R}^N \times \mathbb{R}^M_+ \setminus \mathcal{A}_i, \tag{6.2}$$

where $E^+_i := \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^N \times (\mathbb{R}^*_+)^M : \widetilde{x}^i \geq f_N^{-1}(\sum_{j=1}^M a_{ij} y^j)\}$, $E^-_i := \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^N \times (\mathbb{R}^*_+)^M : \widetilde{x}^i \leq -f_N^{-1}(\sum_{j=1}^M a_{ij} y^j)\}$, and

$$E^+_{i,1} = \left\{ (\boldsymbol{x}, y) \in E^+_i : \sum_{j=1}^{M} a_{ij} y^j \geq \tilde{x}^i + x_0 \right\}, \quad E^+_{i,2} = E^+_i / E^+_{i,1}, \tag{6.3}$$

$$E^-_{i,1} = \left\{ (\boldsymbol{x}, y) \in E^-_i : \sum_{j=1}^{M} a_{ij} y^j \geq -\tilde{x}^i - x_0 \right\}, \quad E^-_{i,2} := E^-_i / E^-_{i,1}, \tag{6.4}$$

and $\{Q_i\}_{i=1}^N$ are convex partitions such that $Q_i \cap Q_j = (E^+_i \cup E^-_i) \cap (E^+_j \cup E^-_j) \cap \partial \mathcal{W}_{NE}$ for $i \neq j$. We then define

$$\Pi(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \left(\left(\boldsymbol{x}^{-i}, x_+^i + \frac{\sum_{k\neq i} x^k}{N-1}\right)\boldsymbol{y}_+^1\right) & \text{if} \quad (\boldsymbol{x},\boldsymbol{y}) \in Q_i \cap E_{i,1}^+, \\ \left(\left(\boldsymbol{x}^{-i}, x^i - \sum_{q=1}^M a_{iq}y^q, \boldsymbol{y}_+^2\right)\right. & \text{if} \quad (\boldsymbol{x},\boldsymbol{y}) \in Q_i \cap E_{i,2}^+, \\ \left(\left(\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} + x_-^i\right), \boldsymbol{y}_-^1\right) & \text{if} \quad (\boldsymbol{x},\boldsymbol{y}) \in Q_i \cap E_{i,1}^-, \\ \left(\left(\boldsymbol{x}^{-i}, x^i + \sum_{q=1}^M a_{iq}y^q\right), \boldsymbol{y}_-^2\right) & \text{if} \quad (\boldsymbol{x},\boldsymbol{y}) \in Q_i \cap E_{i,2}^-, \end{cases} \tag{6.5}$$

in which the threshold function $f_N(\cdot)$ is defined in (4.12)–(4.14), x_+^i is the unique positive root such that $z - f_N(z) = \tilde{x}^i - y^i$ when $\tilde{x}^i \geq f_N^{-1}(y^i)$, and x_-^i is the unique negative root such that $z + \tilde{f}_N(z) = \tilde{x}^i + y^i$ when $\tilde{x}^i \leq -f_N^{-1}(y^i)$. Here $\boldsymbol{y}_+^1 \in \mathbb{R}_+^M$ with the jth component being $(\boldsymbol{y}_+^1)_j = y^j - \frac{a_{ij}y^j}{\sum_{q=1}^M a_{iq}y^q}(\sum_{q=1}^M a_{iq}y^q - f_N(x_+^i))$, $\boldsymbol{y}_+^2 \in \mathbb{R}_+^M$ with the jth component being $(\boldsymbol{y}_+^2)_j = y^j - a_{ij}y^j$, $\boldsymbol{y}_-^1 \in \mathbb{R}_+^M$ with the kth component being $(\boldsymbol{y}_-^1)_j = y^j - \frac{a_{ij}y^j}{\sum_{q=1}^M a_{iq}y^q}(\sum_{q=1}^M a_{iq}y^q - \tilde{f}_N(x_-^i))$, and $\boldsymbol{y}_-^2 \in \mathbb{R}_+^M$ with the jth component being $(\boldsymbol{y}_-^2)_j = y^j - a_{ij}y^j$. Note that $\Pi(\cdot)$ translates $(\boldsymbol{x},\boldsymbol{y})$ to the boundary of $E_{i,1}^+$, i.e., $\partial E_{i,1}^+ := \{(\boldsymbol{x},\boldsymbol{y}) \ : \ \sum_{j=1}^M a_{ij}y^j = f_N^{-1}(\tilde{x}^i), 0 < \tilde{x}_i \leq x_0\}$ when $(\boldsymbol{x},\boldsymbol{y}) \in Q_i \cap E_{i,1}^+$, and to $\{(\boldsymbol{x},\boldsymbol{y}) \ : \ a_{ij}y^j = 0 \text{ for all } j = 1,2,\ldots,M\}$ when $(\boldsymbol{x},\boldsymbol{y}) \in Q_i \cap E_{i,2}^+$, both along the direction $(0,\ldots,-1,\ldots 0; -\frac{a_{i1}y^1}{\sum_{j=1}^M a_{ij}y^j},\ldots,-\frac{a_{iM}y^M}{\sum_{j=1}^M a_{ij}y^j}) \in \mathbb{R}^N \times \mathbb{R}_+^M$ with the ith component being -1. Denote

$$\mathcal{W}_{NE} := \left\{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}_+^M : |\widetilde{x}^i| < f_N^{-1}\left(\sum_{j=1}^M a_{ij}y^j\right), \ 1 \leq i \leq N\right\} \cup \{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}_+^M : \boldsymbol{y} = 0\}, \tag{6.6}$$

and assume the partition $\{Q_i\}_{i=1}^N$ satisfies the following assumption:

H3-**C**. For any $(\boldsymbol{x},\boldsymbol{y}) \in \cup_i \mathcal{A}_i, \quad \Pi(\boldsymbol{x},\boldsymbol{y}) \in \overline{\mathcal{W}_{NE}}$.

Condition H3-**C** implies that if $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_i$, then the dynamics will be in region $\overline{\mathcal{W}_{NE}}$ after player i's control.

From the analysis in sections 4 and 5 and the "guess" that the control policy of player i only depends on $(\boldsymbol{x}, \sum_{j=1}^M a_{ij}y^j)$ when in $\mathcal{W}_{-i}$, we get for $|\widetilde{x}^i| < f_N^{-1}(\sum_{j=1}^M a_{ij}y^j)$,

$$v^i(\boldsymbol{x},\boldsymbol{y}) = p_N(\widetilde{x}^i) + A_N\left(\sum_{j=1}^M a_{ij}y^j\right)\cosh\left(\widetilde{x}^i\alpha_N\right) \tag{6.7}$$

is a solution to (HJB-C), where $p_N(\cdot)$ is defined by (4.10), and $A_N(\cdot)$ defined by (4.13).

The next step is to construct the controlled process $(\boldsymbol{X},\boldsymbol{Y})$ corresponding to the HJB solution (6.7). Note that $\mathcal{W}_{NE}$ is an unbounded domain in $\mathbb{R}^{2N}$ with $2N$ boundaries. For $i = 1,2,\ldots,N$, define the $2N$ faces of $\mathcal{W}_{NE}$ as $F_i = \{(\boldsymbol{x},\boldsymbol{y}) \in \partial\mathcal{W}_{NE} \mid (\boldsymbol{x},\boldsymbol{y}) \in \partial E_i^+\}$ and $F_{i+N} = \{(\boldsymbol{x},\boldsymbol{y}) \in \partial\mathcal{W}_{NE} \mid (\boldsymbol{x},\boldsymbol{y}) \in \partial E_i^-\}$. The normal direction on each face is given by $\boldsymbol{n}_i = c_i(\frac{1}{N-1},\ldots,-1,\ldots,\frac{1}{N-1};(f_N^{-1})'(\sum_{j=1}^M a_{ij}y^j) a_{i1},\ldots,(f_N^{-1})' \ (\sum_{j=1}^M a_{ij}y^j)a_{iM})$ and $\boldsymbol{n}_{N+i} = c_{N+i}(-\frac{1}{N-1},\ldots,1,\ldots,-\frac{1}{N-1}; (f_N^{-1})'(\sum_{j=1}^M a_{ij}y^j)a_{i1},\ldots,(f_N^{-1})'(\sum_{j=1}^M a_{ij}y^j)a_{iM})$ with the ith component being ± 1

and c_i and c_{N+i} the normalizing constants such that $\|\boldsymbol{n}_i\|= \|\boldsymbol{n}_{N+i}\|= 1$. Furthermore, denote the reflection direction on each face as $\boldsymbol{r}_i = c_i'(0,\dots,-1,\dots 0; -\frac{a_{i1}y^1}{\sum_{j=1}^M a_{ij}y^j},\dots, -\frac{a_{iM}y^M}{\sum_{j=1}^M a_{ij}y^j})$ and $\boldsymbol{r}_{N+i} = c_{N+i}'(0,\dots,1,\dots 0; -\frac{a_{i1}y^1}{\sum_{j=1}^M a_{ij}y^j},\dots, -\frac{a_{iM}y^M}{\sum_{j=1}^M a_{ij}y^j})$ with the i^{th} component to be ± 1. c_i' and c_{N+i}' are normalizing constants such that $\|\boldsymbol{r}_i\|= \|\boldsymbol{r}_{N+i}\|= 1$. The NE strategy is defined as follows.

Case 1: $(\boldsymbol{X}_{0-},\boldsymbol{Y}_{0-}) = (\boldsymbol{x},\boldsymbol{y}) \in \overline{\mathcal{W}_{NE}}$. One can check that $\mathcal{W}_{NE}$ defined in (6.6) and $\{\boldsymbol{r}_i\}_{i=1}^{2N}$ defined above satisfy assumptions A1–A5. Therefore, there exists a weak solution to the Skorokhod problem with data $(\mathcal{W}_{NE},\{\boldsymbol{r}_i\}_{i=1}^{2N},\boldsymbol{b},\boldsymbol{\sigma},\boldsymbol{x}\in \overline{\mathcal{W}_{NE}})$. See Appendix A of the online supplementary material for the satisfiability of A1–A5.

Case 2: $(\boldsymbol{X}_{0-},\boldsymbol{Y}_{0-}) = (\boldsymbol{x},\boldsymbol{y}) \notin \overline{\mathcal{W}_{NE}}$. There exists $i \in \{1,\dots,N\}$ such that $(\boldsymbol{X}_{0-},\boldsymbol{Y}_{0-}) \in \mathcal{A}_i$. (1) If $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_+^i \cap E_{i,1}^+$, then player i will move immediately from $X_{0-}^i = x^i$ to $X_0^i = x_+^i + \frac{\sum_{k\neq i} x^k}{N-1}$ at time 0, where x_+^i is the unique positive root such that $z - f_N(z) = \tilde{x}^i - (\sum_{q=1}^M a_{iq}y^q)$. This will reduce the resources from $\boldsymbol{Y}_{0-} = \boldsymbol{y}$ to $\boldsymbol{Y}_0 = \boldsymbol{y}_+$ with the jth component of $\boldsymbol{y}_+$ being $(\boldsymbol{y}_+)_j = y^j - \frac{a_{ij}y^j}{\sum_{q=1}^M a_{iq}y^q}(\sum_{q=1}^M a_{iq}y^q - f_N(x_+^i)) \geq 0$. Other players' dynamics remain unchanged, i.e., $X_0^k = X_{0-}^k = x^k$ for $k \neq i$ and $1 \leq k \leq N$. By assumption H3-**C**, we have $(\boldsymbol{X}_0,\boldsymbol{Y}_0) = ((\boldsymbol{x}^{-i}, x_+^i + \frac{\sum_{k\neq i} x^k}{N-1}),\boldsymbol{y}_+) = \Pi(\boldsymbol{X}_{0-},\boldsymbol{Y}_{0-}) \in \overline{\mathcal{W}_{NE}}$. (2) If $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_i^+ \cap E_{i,2}^+$, then player i will move immediately from $X_{0-}^i = x^i$ to $X_0^i = x^i - \sum_{q=1}^M a_{iq}y^q$ and resource j is changed from $Y_{0-}^j = y^j$ to $Y_0^j = y^j - a_{ij}y^j$ at time 0. Other players' dynamics remain unchanged, i.e., $X_0^k = X_{0-}^k = x^k$ for $k \neq i$ and $1 \leq k \leq N$. Under assumption H3-**C**, we have $(\boldsymbol{X}_0,\boldsymbol{Y}_0) = \Pi(\boldsymbol{X}_{0-},\boldsymbol{Y}_{0-}) \in \overline{\mathcal{W}_{NE}}$. (3) Similarly, if $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_i^- \cap E_{i,1}^-$, then player i will move immediately from $X_{0-}^i = x^i$ to $X_0^i = x_-^i + \frac{\sum_{k\neq i} x^k}{N-1}$ at time 0, where x_-^i is the unique negative root such that $z + \tilde{f}_N(z) = \tilde{x}^i + (\sum_{q=1}^M a_{iq}y^q)$. This changes the resources from $\boldsymbol{Y}_{0-} = \boldsymbol{y}$ to $\boldsymbol{Y}_0 = \boldsymbol{y}_-$, where the jth component of $\boldsymbol{y}_-$ is $(\boldsymbol{y}_-)_j = y^j - \frac{a_{ij}y^j}{\sum_{q=1}^M a_{iq}y^q}(\sum_{q=1}^M a_{iq}y^q - \tilde{f}_N(x_-^i)) \geq 0$. Other players' dynamics remain unchanged at time 0, i.e., $X_0^k = X_{0-}^k = x^k$ for $k \neq i$ and $1 \leq k \leq N$. By assumption H3-**C**, we have $(\boldsymbol{X}_0,\boldsymbol{Y}_0) = ((\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} + x_-^i),\boldsymbol{y}_-) = \Pi(\boldsymbol{X}_{0-},\boldsymbol{Y}_{0-}) \in \overline{\mathcal{W}_{NE}}$. (4) If $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_i^+ \cap E_{i,2}^+$, then player i will move immediately from $X_{0-}^i = x^i$ to $X_0^i = x^i + \sum_{q=1}^M a_{iq}y^q$ and resource j is reduced from $Y_{0-}^j = y^j$ to $Y_0^j = y^j - a_{ij}y^j$ at time 0. Other players' dynamics remain unchanged at time 0. By assumption H3-**C**, we have $(\boldsymbol{X}_0,\boldsymbol{Y}_0) = \Pi(\boldsymbol{X}_{0-},\boldsymbol{Y}_{0-}) \in \overline{\mathcal{W}_{NE}}$.

The NE for the N-player game (2.8) with constraint $\boldsymbol{C}$ is stated as follows.

THEOREM 6.1 (NE for the N-player game $\boldsymbol{C}$). *Assume* H1′–H2′ *and* H3-**C**. *Define* $u^i \in \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ *as*

(6.8)
$$u^i(\boldsymbol{x},y) = \begin{cases} p_N(\tilde{x}^i) + A_N(y)\cosh\left(\alpha_N \tilde{x}^i\right) & \text{if } |\tilde{x}^i|\leq f_N^{-1}(y) \text{ and } y=0,\\ u^i\left(\boldsymbol{x}^{-i}, x_+^i + \frac{\sum_{k\neq i} x^k}{N-1}, f_N(x_+^i)\right) & \text{if } \tilde{x}^i > f_N^{-1}(y) \text{ and } y \geq \tilde{x}^i + x_0,\\ u^i\left(\boldsymbol{x}^{-i}, x^i - y, 0\right) & \text{if } \tilde{x}^i > f_N^{-1}(y) \text{ and } y < \tilde{x}^i + x_0,\\ u^i\left(\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} + x_-^i, \tilde{f}_N(x_-^i)\right) & \text{if } \tilde{x}^i < -f_N^{-1}(y) \text{ and } y \geq -\tilde{x}^i - x_0,\\ u^i\left(\boldsymbol{x}^{-i}, x^i + y, 0\right) & \text{if } \tilde{x}^i < -f_N^{-1}(y) \text{ and } y < -\tilde{x}^i - x_0, \end{cases}$$

and define $v^i : \mathbb{R}^N \times \mathbb{R}_+^M \to \mathbb{R}$ *as*

(6.9)
$$v^i(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} u^i\left(\boldsymbol{x}, \sum_{j=1}^M a_{ij} y^j\right) & \text{if } (\boldsymbol{x},\boldsymbol{y}) \in \overline{\mathcal{W}_{-i}}, \\ v^i\left(\boldsymbol{x}^{-j}, x_+^j + \frac{\sum_{k\neq j} x^k}{N-1}, \boldsymbol{y}_+^1\right) & \text{if } (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_j^+ \cap E_{j,1}^+ \text{ for } j \neq i, \\ v^i\left(\boldsymbol{x}^{-j}, x^j - \left(\sum_{q=1}^M a_{jq} y^q\right), \boldsymbol{y}_+^2\right) & \text{if } (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_j^+ \cap E_{j,2}^+ \text{ for } j \neq i, \\ v^i\left(\boldsymbol{x}^{-j}, \frac{\sum_{k\neq j} x^k}{N-1} + x_-^j, \boldsymbol{y}_-^1\right) & \text{if } (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_j^- \cap E_{j,1}^- \text{ for } j \neq i, \\ v^i\left((\boldsymbol{x}^{-j}, x^j + \left(\sum_{q=1}^M a_{jq} y^q\right), \boldsymbol{y}_-^2\right) & \text{if } (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_j^- \cap E_{j,2}^- \text{ for } j \neq i, \end{cases}$$

where $\mathcal{A}_i$ *and* $\mathcal{W}_i$ *are given in* (6.2), $E_{i,1}^{\pm}$ *and* $E_{i,2}^{\pm}$ *are given in* (6.3)–(6.4) *with* $f_N(\cdot)$ *defined by* (4.12)–(4.14), *and* $\tilde{f}_N(x) = f_N(-x)$ *for* $x < 0$*;* $\widetilde{x}^i$ *is defined by* (4.2), *and* $A_N(\cdot)$ *is defined by* (4.13)*;* x_+^i *in* (6.8) *is the unique positive root of* $z - f_N(z) = \widetilde{x}^i - y$ *if* $\tilde{x}^i \geq f_N^{-1}(y)$, *and* x_-^i *is the unique negative root of* $z + \tilde{f}_N(z) = \widetilde{x}^i + y$ *if* $\tilde{x}^i < -f_N^{-1}(y)$*;* x_+^j *in* (6.9) *is the unique positive root of* $z - f_N(z) = \widetilde{x}^j - \sum_{k=1}^M a_{jk} y^k$ *if* $\tilde{x}^j \geq f_N^{-1}(\sum_{q=1}^M a_{jq} y^q)$, *and* x_-^j *is the unique negative root of* $z + \tilde{f}_N(z) = \widetilde{x}^j + \sum_{k=1}^M a_{jk} y^k$ *if* $\tilde{x}^j < -f_N^{-1}(\sum_{q=1}^M a_{jq} y^q)$*; the kth component of* $\boldsymbol{y}_+^1$ *in* (6.9) *is* $(\boldsymbol{y}_+^1)_k = y^k - \frac{a_{jk} y^k}{\sum_{q=1}^M a_{jq} y^q}(\sum_{q=1}^M a_{jq} y^q - f_N(x_+^j))$, *and the kth component of* $\boldsymbol{y}_-^1$ *is* $(\boldsymbol{y}_-^1)_k = y^k - \frac{a_{jk} y^k}{\sum_{q=1}^M a_{jq} y^q}(\sum_{q=1}^M a_{jq} y^q - \tilde{f}_N(x_-^j))$*; and finally the kth component of* $\boldsymbol{y}_+^2$ *in* (6.9) *is* $(\boldsymbol{y}_+^2)_k = y^k - a_{jk} y^k$, *and the kth component of* $\boldsymbol{y}_-^2$ *is* $(\boldsymbol{y}_-^2)_k = y^k - a_{jk} y^k$. *Then* v^i *is the value associated with a NEP* $\boldsymbol{\xi}^* = (\xi^{1*}, \ldots, \xi^{N*})$. *That is,* $v^i(\boldsymbol{x},\boldsymbol{y}) = J_C^i(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi}^*)$. *Moreover, the controlled process* $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ *under* $\boldsymbol{\xi}^*$ *is a solution to a Skorokhod problem as described in* Case 1 *if* $(\boldsymbol{x},\boldsymbol{y}) \in \overline{\mathcal{W}_{NE}}$ *and described as* Case 2 *if* $(\boldsymbol{x},\boldsymbol{y}) \notin \overline{\mathcal{W}_{NE}}$.

The proof of Theorem 6.1 is similar to that of Theorem 4.3 and hence omitted.

7. Comparing Games $\boldsymbol{C_p}$, $\boldsymbol{C_d}$, and $\boldsymbol{C}$.

In this section, we compare the games $\boldsymbol{C_p}$, $\boldsymbol{C_d}$, and $\boldsymbol{C}$. We will first compare their game values and discuss their economic implications. We will then discuss their difference in terms of the NEP. Finally, we discuss their perspective NEs in the framework of controlled rank-dependent SDEs.

To make the games comparable, let us assume $y = \sum_{j=1}^N y^j$. Let us also consider a special sharing game $\boldsymbol{C_s}$ which can be connected with both $\boldsymbol{C_d}$ and $\boldsymbol{C_p}$:

$\boldsymbol{C_s}$. $M = N$ and $a_{ii} = 1$ for $i = 1, 2, \ldots, N$.

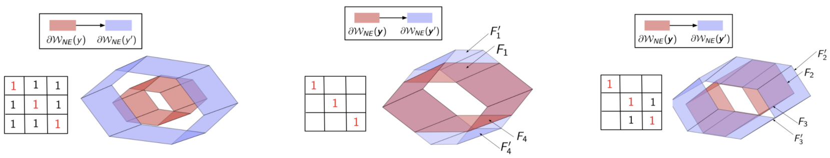
7.1. Pooling, dividing, and sharing.

Denote the game value and waiting region for each player i as $v_{C_p}^i$ and $\mathcal{W}_i^{C_p}$, respectively, for game $\boldsymbol{C_p}$. Similar notations are defined for $\boldsymbol{C_d}$ and $\boldsymbol{C_s}$.

PROPOSITION 7.1 (game values comparison). *Assume* H1′–H2′. *For each* $(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}_+^N$, *denote* $y = \sum_{i=1}^N y^i$. *If* $(\boldsymbol{x}, y) \in \mathcal{W}_i^{C_p}$ *and* $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{W}_i^{C_d} \cap \mathcal{W}_i^{C_s}$, *then,*

$$v_{C_p}^i(\boldsymbol{x}, y) \leq v_{C_s}^i(\boldsymbol{x}, y) \leq v_{C_d}^i(\boldsymbol{x},\boldsymbol{y}), \ i = 1, 2, \ldots, N.$$

The proof of Proposition 7.1 is straightforward and hence omitted. Details are provided in the online supplementary material. This result has a clear economic



(a) $\boldsymbol{C_p}$ (b) $\boldsymbol{C_d}$ (c) $\boldsymbol{C}$

Fig. 3. *Comparison of projected evolving boundaries for* $\boldsymbol{C_p}$, $\boldsymbol{C_d}$, $\boldsymbol{C}$ *when* $N = 3$.

interpretation. In a stochastic game where players have the option to share resources, versus the possibility to divide resources in advance, sharing will have a lower cost than dividing. Pooling yields the lowest cost for each player.

Define the projected common waiting region $\mathcal{W}_{NE}(\boldsymbol{y}) := \{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times (\mathbb{R}_+^*)^M : |\widetilde{x}^i| < f_N^{-1}(\sum_{j=1}^M a_{ij} y^j) \text{ for } 1 \le i \le N\} \cup \{\boldsymbol{y} = 0\}$, for any fixed resource level $\boldsymbol{y}$. Then $\mathcal{W}_{NE}(\boldsymbol{y})$ is a polyhedron with $2N$ boundary faces. Figure 3(a) shows a pooling game $\boldsymbol{C_p}$. After one player exercises controls, all the faces of the boundary move. Figure 3b corresponds to a dividing game $\boldsymbol{C_d}$. After player i exercises controls, her faces of F_i and F_{i+N} move. Here $i = 1, N = 3$. For a sharing game $\boldsymbol{C}$, shown in Figure 3c, after one player exercises her control, the faces of the players who are connected with her will move, while the faces for other players remain unchanged. Here $i = 2$ and players two and three are connected.

7.2. NEs for the games and controlled rank-dependent SDEs. In the previous sections, the controlled dynamics is constructed directly via the reflected Brownian motion. This class of SDEs can also be cast in the framework of rank-dependent SDEs. Indeed, the controlled dynamics of NE in the action regions of the N-player can be written as *controlled rank-dependent SDEs*:

$$dX_t^i = \sum_{j=1}^N 1_{F^i(\boldsymbol{X}_t,\boldsymbol{Y}_t)=F^{(j)}(\boldsymbol{X}_t,\boldsymbol{Y}_t)} \left(\delta_j dt + \sigma_j dB_t^j + d\xi_t^{j,+} - d\xi_t^{j,-}\right),$$

$$dY_t^j = -\sum_{i=1}^N \frac{a_{ij} Y_{s-}^j}{\sum_{k=1}^M a_{ik} Y_{s-}^k} d\check{\xi}_s^i$$

with $(\xi^{i,+}, \xi^{i,-})$ the controls, $F^i : \mathbb{R}^N \times \mathbb{R}_+^M \to \mathbb{R}$ a rank function depending on both $\boldsymbol{X}$ and $\boldsymbol{Y}$, $F^{(1)} \le \cdots \le F^{(N)}$ the order statistics of $(F^i)_{1 \le i \le N}$, and $\delta_i \in \mathbb{R}$, $\sigma_i \ge 0$.

In game $\boldsymbol{C_p}$, the controlled dynamics in the action regions satisfies the SDEs with $F_{C_p}^i(\boldsymbol{x},\boldsymbol{y}) = |x^i - \frac{\sum_{j \ne i} x^j}{N-1}|$, $\delta_i = 0$, and $\sigma_i = 0$ for each $i = 1, \dots N$, and $\xi^{i,\pm} = 0 \quad$ for each $i = 1, \dots, N-1$ and $\xi^{N,\pm} \ne 0$.

In game $\boldsymbol{C_d}$, $F_{C_d}^i(\boldsymbol{x},\boldsymbol{y}) = |x^i - \frac{\sum_{j \ne i} x^j}{N-1} - f_N^{-1}(y^i)|$. For the general game $\boldsymbol{C}$, the controlled process in the action regions is governed by the rank-dependent dynamics with $F_C^i(\boldsymbol{x},\boldsymbol{y}) = |x^i - \frac{\sum_{j \ne i} x^j}{N-1} - f_N^{-1}(\sum_{j=1}^M a_{ij} y^j)|$ with f_N a threshold function defined in (4.12)–(4.14) and δ_i, σ_i, and $\xi^{i,\pm}$ satisfying the same condition as before.

Note that the special case without controls, i.e., $F^i(\boldsymbol{x},\boldsymbol{y}) = x^i$ and $\xi^{i,\pm} = 0$, corresponds to the *rank-dependent SDEs*. In particular, the rank-dependent SDEs with $\delta_1 = 1$, $\delta_2 = \cdots \delta_N = 0$ are known as the *Atlas model*. To the best of our knowledge, rank-dependent SDEs with additional controls or a general rank function F^i have not been studied before. There are various aspects including uniqueness and sample path properties that await further investigation and we leave them to interested readers.

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