

MCKEAN–VLASOV EQUATIONS INVOLVING HITTING TIMES: BLOW-UPS AND GLOBAL SOLVABILITY

BY ERHAN BAYRAKTAR^{1,a}, GAOYUE GUO^{2,b}, WENPIN TANG^{3,c} AND YUMING PAUL ZHANG^{4,d}

¹Department of Mathematics, University of Michigan, ^aerhan@umich.edu

²Laboratoire MICS and CNRS FR-3487, Université Paris-Saclay CentraleSupélec, ^bgaoyue.guo@centralesupelec.fr

³Department of Industrial Engineering and Operations Research, Columbia University, ^cwt2319@columbia.edu

⁴Department of Mathematics, University of California, San Diego, ^dyzhangpaul@ucsd.edu

This paper is concerned with the analysis of blow-ups for two McKean–Vlasov equations involving hitting times. Let $(B(t); t \geq 0)$ be standard Brownian motion, and $\tau := \inf\{t \geq 0 : X(t) \leq 0\}$ be the hitting time to zero of a given process X . The first equation is $X(t) = X(0-) + B(t) - \alpha \mathbb{P}(\tau \leq t)$. We provide a simple condition on α and the distribution of $X(0-)$ such that the corresponding Fokker–Planck equation has no blow-up, and thus the McKean–Vlasov dynamics is well defined for all time $t \geq 0$. Our approach relies on a connection between the McKean–Vlasov equation and the supercooled Stefan problem, as well as several comparison principles. The second equation is $X(t) = X(0-) + \beta t + B(t) + \alpha \ln \mathbb{P}(\tau > t)$, $t \geq 0$, whose Fokker–Planck equation is nonlocal. We prove that for $\beta > 0$ sufficiently large and α no greater than a sufficiently small positive constant, there is no blow-up and the McKean–Vlasov dynamics is well defined for all time $t \geq 0$. The argument is based on a new transform, which removes the nonlocal term, followed by a relative entropy analysis.

1. Introduction and main results. Complex systems are central to the scientific modeling of real world phenomena. A challenge in mathematical modeling is to provide reasonably simple frameworks to capture the collective behaviors of individuals with intricate interactions. One famous example is the McKean–Vlasov equations, which were considered by Kac [23] in the context of statistical physics, and were further developed by McKean [30] to study weakly interacting particles. The McKean–Vlasov equations have proved to be a powerful tool for modeling the mean field behavior of disordered systems, with applications including the dynamics of granular media [4, 5], mathematical biology [6, 24], economics and social networks [10, 22], and deep neural networks [31, 34]. There have been a rich body of works on McKean–Vlasov equations, see [8, 9] for a detailed exposition.

In this paper, we are concerned with a class of generalized McKean–Vlasov equations which involve hitting times as boundary penalties. These equations take the general form

$$(1.1) \quad \begin{cases} X(t) = X(0-) + \beta t + B(t) + f(s(t)), & t \geq 0, \\ \tau := \inf\{t \geq 0 : X(t) \leq 0\}, \\ s(t) := \mathbb{P}(\tau \leq t), \end{cases}$$

where $X(0-)$ has a distribution supported on $(0, \infty)$, $\beta \in \mathbb{R}$ is the drift, $(B(t); t \geq 0)$ is standard Brownian motion, and $f : [0, 1) \rightarrow \mathbb{R}$ is a feedback function. While our approaches may be used to study general cases, we focus on the following two special scenarios.

Received July 2022; revised May 2023.

MSC2020 subject classifications. 35K61, 60H30.

Key words and phrases. Blow-ups, comparison principle, Fokker–Planck equations, generalized solution, entropy, hitting times, McKean–Vlasov equations, self-similar solution, Stefan problem.

1. $\beta = 0$ and $f(x) = -\alpha x, \alpha > 0$: The equation (1.1) specializes to

$$(1.2) \quad \begin{cases} X(t) = X(0-) + B(t) - \alpha \mathbb{P}(\tau \leq t), & t \geq 0, \\ \tau := \inf\{t \geq 0 : X(t) \leq 0\}. \end{cases}$$

This model was originated in the study of the integrate-and-fire mechanism in neuroscience, from both probability aspects [15, 16] and PDE perspectives [7, 11, 12]. It also arose as a toy model to study the mean field behavior of contagious financial networks [21]. These works mainly dealt with the well-posedness of the McKean–Vlasov dynamics (1.2). By letting $p(t, \cdot)$ be the subprobability density of $X(t)1_{\{\tau > t\}}$ and $N(t) := \partial_t \mathbb{P}(\tau \leq t)$, the corresponding Fokker–Planck equation is

$$(1.3) \quad \begin{cases} p_t = \frac{1}{2} p_{xx} + \alpha N(t) p_x & \text{in } [0, T) \times (0, \infty), \\ N(t) = \frac{1}{2} p_x(t, 0), \quad p(t, 0) = 0 & \text{for } t \in [0, T), \\ p(0, x) = p_0(x) & \text{for } x \in [0, \infty), \end{cases}$$

where $p_0(x)$ is the probability density of $X(0-)$. As was shown in [11], the negative feedback $\alpha \leq 0$ is classical, and there is a unique smooth solution to (1.3) for all $t \geq 0$. The positive feedback $\alpha > 0$ is more subtle [7, 21]: $s(t) := \mathbb{P}(\tau \leq t)$ may not be absolutely continuous, and there may exist $T_* > 0$ such that $N(T_*) = \infty$. Such T_* is called a blow-up, which is the main obstacle to analyze the Fokker–Planck equation (1.3), and study the well-posedness of the McKean–Vlasov dynamics (1.2). To work around the blow-ups, [16] proposed the notion of a “physical solution” satisfying $s(t) - s(t-) = \inf\{x \geq 0 : \mathbb{P}(X(t-) \in (0, \alpha x)) < x\}$ to the McKean–Vlasov dynamics (1.2), and proved the global existence by a particle system approximation. A recent breakthrough [17] connected the Fokker–Planck equation (1.3) to the supercooled Stefan problem, and as a byproduct the uniqueness of the physical solution is proved in the presence of blow-ups. See also [2, 14, 28] for related developments.

2. $f(x) = \alpha \ln(1 - x), \alpha > 0$: The equation (1.1) specializes to

$$(1.4) \quad \begin{cases} X(t) = X(0-) + \beta t + B(t) + \alpha \ln \mathbb{P}(\tau > t), & t \geq 0, \\ \tau := \inf\{t \geq 0 : X(t) \leq 0\}. \end{cases}$$

Similar to [21], this model was proposed in [32] to study the systemic risk of default financial networks. The nonlinearity of “ln” comes from the assumption that after k banks default at time t , the value of each remaining bank is reduced by a factor of $(1 - \frac{k}{\# \text{ banks at time } t-})^{-\alpha}$. Such a phenomenon is called a default cascade. By letting $q(t, \cdot)$ be the subprobability density of $X(t)1_{\{\tau > t\}}$ and $\lambda(t) := \partial_t \ln \mathbb{P}(\tau > t)$, the corresponding Fokker–Planck equation is nonlocal:

$$(1.5) \quad \begin{cases} q_t = \frac{1}{2} q_{xx} - (\alpha \lambda(t) + \beta) q_x & \text{in } [0, T) \times (0, \infty), \\ \lambda(t) = -\frac{1}{2} \frac{q_x(t, 0)}{\int_0^\infty q(t, y) dy}, \quad q(t, 0) = 0 & \text{on } [0, T), \\ q(0, x) = q_0(x) & \text{on } [0, \infty), \end{cases}$$

where $q_0(x)$ is the probability density of $X(0-)$. Note that the equation (1.5) is slightly different from that in [32], Section 3, since we define $\lambda(t)$ as $\partial_t \ln \mathbb{P}(\tau > t)$ instead of $\alpha \partial_t \ln \mathbb{P}(\tau > t)$. There may also exist a blow-up T_* such that $\|\lambda\|_{L^2[0, T_*]} := \int_0^{T_*} \lambda^2(t) dt = \infty$. It was proved in [32] that a physical solution exists for all time t . The uniqueness is still open, though it is believable that the arguments in [17] carry over to this setting.

As we have already seen, the main difficulty in analyzing the McKean–Vlasov dynamics involving hitting times arises from the blow-ups. Though [16, 17, 32] proposed the physical solution to overcome this problem, it is still interesting to know whether there is a blow-up or not. This is because the existence of a blow up implies a possible systemic risk event in a financial network. To be more precise, we ask the following question.

QUESTION. Under what conditions on the distribution of X_{0-} is there no blow-up in the Fokker–Planck equations (1.3) and (1.5) respectively?

If there is no blow-up, the physical solution coincides with a smooth solution possibly in some weak sense. We simply say that the McKean–Vlasov dynamics is (well-)defined for all $t \geq 0$ if the corresponding Fokker–Planck equation does not exhibit blow-ups, and is hence defined for all time in the classical sense. In [15, 21], no blow-up conditions for the Fokker–Planck equation (1.3) have been studied, which assure that the McKean–Vlasov dynamics (1.2) is defined for all time $t \geq 0$. However, these conditions seem to be obscure, and are not easy to check. No blow-up conditions for the Fokker–Planck equation (1.5) have yet been explored, and it was conjectured in [32] that there is no blow-up if α is sufficiently small.

In this paper, we provide a simple criterion on the distribution of X_{0-} under which the Fokker–Planck equation (1.3) does not have any blow-up. We also study the problem of blow-ups for the Fokker–Planck equation (1.5), resolving the aforementioned conjecture. To state the results, we need the following definition of weak and generalized solutions to (1.3) and (1.5), respectively. Below $L^1_{\text{loc}}([0, T])$ (resp. $L^2_{\text{loc}}([0, T])$) denotes the functions that are locally uniformly L^1 (resp. L^2) in $[0, T]$, and $L^\infty([0, T]; L^1(\mathbb{R}^+))$ denotes

$$\left\{ f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^+ \mid f(t, \cdot) \in L^1(\mathbb{R}^+), \sup_{t \in [0, T]} \|f(t, \cdot)\|_{L^1(\mathbb{R}^+)} < \infty \right\},$$

and $W^{1,2}_2([0, T] \times [0, \infty))$ denotes the Sobolev space $L^2([0, T] \times [0, \infty))$ whose first weak derivative in time and the first two weak derivatives in space belong to $L^2([0, T] \times [0, \infty))$, equipped with the associated Sobolev norm.

DEFINITION 1.1.

1. A pair of functions (p, N) is a weak solution to the Fokker–Planck equation (1.3) in the time interval $[0, T]$ if

$$p \in L^\infty([0, T]; L^1(\mathbb{R}^+)), \quad N \in L^1_{\text{loc}}([0, T]),$$

p, N are nonnegative, and for any test function $\phi(t, x) \in C^\infty([0, T'] \times [0, \infty))$ with $T' < T$ such that $\phi, \phi_t, \phi_x, \phi_{xx} \in L^\infty([0, T'] \times [0, \infty))$, we have

$$\begin{aligned} & \int_0^{T'} \int_0^\infty p(t, x) \left[-\phi_t(t, x) + \alpha N(t) \phi_x(t, x) - \frac{1}{2} \phi_{xx}(t, x) \right] dx dt \\ &= - \int_0^{T'} N(t) \phi(t, 0) dt + \int_0^\infty p_0(x) \phi(0, x) dx - \int_0^\infty p(T', x) \phi(T', x) dx. \end{aligned}$$

2. [26, 32] A pair of functions (q, λ) is a generalized solution to the Fokker–Planck equation (1.5) in the time interval $[0, T]$ if

$$\lambda \in L^2_{\text{loc}}([0, T]),$$

and for $T' < T$, the unique solution to the equation

$$q_t = \frac{1}{2} q_{xx} - (\alpha \lambda(t) + \beta) q_x, \quad q(t, 0) = 0, \quad q(0, x) = q_0(x),$$

in $W^{1,2}_2([0, T'] \times [0, \infty))$ satisfies $\lambda(t) = -\frac{1}{2} \frac{q_x(t, 0)}{\int_0^\infty q(t, y) dy}$ for almost every $t \in [0, T']$.

In the above definition, a generalized solution to the equation (1.5) is more restrictive than a weak solution to the equation (1.3), and it requires the uniqueness along with the existence. This complication is due to the nonlocal term “ $\int_0^\infty q(t, y) dy$ ” in the Fokker–Planck equation (1.5). Our first result provides sufficient conditions for the Fokker–Planck equations (1.3) and (1.5) to exhibit blow-ups, which is in a similar spirit to [7], Theorem 2.2.

PROPOSITION 1.1.

1. If there exists $\mu > 0$ such that

$$(1.6) \quad \mu\alpha \int_0^\infty e^{-\mu x} p_0(x) dx \geq 1,$$

then there is no weak solution to the Fokker–Planck equation (1.3) for all time $t \geq 0$. In this case, the solution can only exist before time

$$(1.7) \quad T := \frac{2}{\mu^2} \ln \frac{\int_0^\infty p_0(x) dx}{\int_0^\infty e^{-\mu x} p_0(x) dx}.$$

In particular, if $\alpha > \min_{\mu>0} \{(\mu \int_0^\infty e^{-\mu x} p_0(x) dx)^{-1}, 2 \int_0^\infty x p_0(x) dx\}$, then there is no weak solution to the Fokker–Planck equation (1.3) for all time $t \geq 0$.

2. If there exists a positive number $\mu > 2\beta$ such that

$$(1.8) \quad (1 + \alpha\mu) \int_0^\infty e^{-\mu x} q_0(x) dx \geq \int_0^\infty q_0(x) dx,$$

then there is no generalized solution to the Fokker–Planck equation (1.5) for all time $t \geq 0$. In this case, the solution can only exist before time

$$(1.9) \quad T := \frac{2}{\mu(\mu - 2\beta)} \ln \left(\frac{\int_0^\infty q_0(x) dx}{\int_0^\infty e^{-\mu x} q_0(x) dx} \right).$$

In view of Lemma 2.3, if we further assume

$$(1.10) \quad \limsup_{x \rightarrow 0^+} p_0(x) < \frac{1}{\alpha} \quad \text{and} \quad \lim_{x \rightarrow \infty} p_0(x) = 0,$$

then the solution blows up ($N(t) \rightarrow \infty$) before time T with T defined by (1.7). By Lemma 3.1, if we assume that $q_0(\cdot) \in W_2^1([0, \infty))$ and $q_0(0) = 0$, then there exists a time $t_{\text{reg}} \in (0, T]$ with T defined by (1.9) such that $\lim_{t \uparrow t_{\text{reg}}} \|\lambda\|_{L^2[0, t]} = \infty$. We emphasize that the physical solution allows the presence of blow ups, and Proposition 1.1 then implies that the first blow up must occur before time T in each Fokker–Planck equation.

The next theorem, which is our main result, gives a simple condition under which there is no blow-up for the Fokker–Planck equation (1.3). Thus, a weak solution to (1.3), which is proved to be a classical one, is defined for all time $t \geq 0$. Consequently, the McKean–Vlasov dynamics (1.2) is defined for all time $t \geq 0$.

THEOREM 1.1. Let p_0 be a probability density supported on $(0, \infty)$ such that (1.10) holds and

$$(1.11) \quad \int_0^x (1 - \alpha p_0(y)) dy > 0 \quad \text{for all } x \in (0, \infty).$$

Assume that the weak solution (p, N) to the Fokker–Planck equation (1.3) with initial data p_0 exists for a short time. Then it exists for all time $t \geq 0$. Moreover, (p, N) satisfies the equation in the classical sense, and for all $t > 0$, and for some $C > 0$ depending only on p_0 we have

$$N(t) \leq C\alpha^{-1} (1 + \alpha^{-1} + (1 + \alpha^{\frac{1}{2}})t^{-\frac{1}{2}} + (\ln t)^2) \quad \text{for all } t > 0.$$

Note that under the assumption (1.10), if we further assume the probability density p_0 to be piecewise continuous, then the weak solution (p, N) exists for a short time. This can be done by the argument in [19], Theorem 1.3.

Note that the condition (1.11) is weaker than $\alpha < \|p_0\|_\infty^{-1}$. Indeed, we do not assume any L^∞ bound on the initial data. It was proved in [27], Theorem 2.2, that if $\alpha < \|p_0\|_\infty^{-1}$, the McKean–Vlasov dynamics (1.2) is pathwise unique. Combining this result with Theorem 1.1, we get the following corollary.

COROLLARY 1.1. *Assume that $\lim_{x \rightarrow \infty} p_0(x) = 0$ and $\alpha < \|p_0\|_\infty^{-1}$. Then the McKean–Vlasov dynamics (1.2) is defined for all time $t \geq 0$, and is pathwise unique.*

Furthermore, we consider the Fokker–Planck equation (1.3) with initial data of form δ_{x_0} (delta mass). Applying Proposition 1.1(1) and Theorem 1.1 yields the following corollary.

COROLLARY 1.2. *Let $\alpha, x_0 > 0$. Let $(p(\cdot, x; x_0), N(\cdot; x_0))$ be a weak solution to the Fokker–Planck equation (1.3) with initial data δ_{x_0} , and assume that $(p(\cdot, x; x_0), N(\cdot; x_0))$ exists for a small time. Then:*

- if $\alpha < x_0$, the solution $p(t, \cdot; x_0)$ exists for all time $t \geq 0$ and $N(t; x_0) < \infty$ for all $t > 0$.
- if $\alpha > 2x_0$, the solution cannot exist for all time. Moreover, there exists $T_{x_0} > 0$ such that $\limsup_{t \rightarrow T_{x_0}} p_x(t, \cdot; x_0) = \infty$.

Now we turn to the Fokker–Planck equation (1.5). Due to the nonlocal term, it seems to be difficult to get a simple criterion for no blow-up. Nevertheless, we are able to show that for any initial data if $\beta > 0$ is sufficiently large, and α is no greater than a sufficiently small positive constant, then a generalized solution to (1.5) and the McKean–Vlasov dynamics (1.4) is well defined for all time $t \geq 0$. This confirms a conjecture in [32], Remark 2.8. Below $W_2^1([0, \infty))$ denotes the Sobolev space of $L^2([0, \infty))$ functions whose first weak derivative belongs to $L^2([0, \infty))$.

THEOREM 1.2. *Let $q_0(\cdot) \in W_2^1([0, \infty))$ with $q_0(0) = 0$, and assume that $q_0^2(x)/x$ is integrable on $(0, 1)$. There exists $C_0 > 0$ depending only on q_0 such that if $\beta \geq C_0$ and $\alpha \leq \frac{1}{C_0}$, then the generalized solution (q, λ) to the Fokker–Planck equation (1.5) with initial data q_0 exists for all time $t \geq 0$. Moreover, for some $C > 0$ depending only on q_0 we have*

$$(1.12) \quad \int_0^t |\lambda(s)|^2 ds \leq C(1+t) \quad \text{for all } t > 0.$$

Note that [32] only considers the case $\alpha > 0$, while Theorem 1.2 extends to all $\alpha \leq 0$ provided that $\beta > 0$ is sufficiently large. As a consequence of (1.12), there exists $C' > 0$ such that $\mathbb{P}(\tau > t) \geq \exp(-C't)$ for all $t > 0$, which gives a lower bound on the tail of the hitting time of the McKean–Vlasov dynamics (1.4). The problem of the uniqueness is more subtle. In our forthcoming paper [3], we prove that for some initial distribution q_0 , the McKean–Vlasov dynamics (1.4) for large β and small α is not unique in distribution.

The main idea to prove Theorem 1.1 and Theorem 1.2 consists of comparing the solution to (1.3) with the self-similar solution to the super-cooled Stefan problem, and comparing the solution to (1.5) with the stationary solution to a transformed equation. In contrast with the fixed-point method used in [15, 21, 32], we rely on comparison principles and relative entropy arguments which are of independent interest.

Organization of the paper. In Section 2, we consider the weak solutions to the equation (1.3). There we prove Theorem 1.1. Section 3 is devoted to the study of the equation (1.5), and Theorem 1.2 is proved.

2. Solutions to the Fokker–Planck equation (1.3). In this section, we study the weak solutions to the Fokker–Planck equation (1.3). The idea of the proof is inspired from [19], Theorem 4.1, regarding the supercooled Stefan problem. Using the transformation (2.4) below, it can be deduced from [19] that if

$$\limsup_{x \rightarrow 0^+} p_0(x) < \frac{1}{\alpha},$$

the solution p to the equation (1.3) exists for a short time, and within the short time $N(t) < \infty$ for $t > 0$. In comparison with [19], our approach is notably different: (1) our existence and regularity results hold for all time; (2) we consider an unbounded domain. To achieve these, technically, we need more delicate comparison principles (Lemma 2.4 and Lemma 2.5).

Section 2.1 presents preliminaries on the super-cooled Stefan problem, and its self-similar solution. In Section 2.2, we provide key comparison lemmas which will be used in the proof of Theorem 1.1. Theorem 1.1 and Corollary 1.2 will be proved in Section 2.3.

2.1. *Preliminaries.* We also need the notion of classical solutions to the equation (1.3). To this end, we assume that the initial data p_0 satisfies

$$(2.1) \quad \begin{aligned} p_0 &\in C^1(\mathbb{R}^+) \cap C([0, \infty)) \cap L^1(\mathbb{R}^+), \\ p_0(0) &= \lim_{x \rightarrow \infty} p_0(x) = \lim_{x \rightarrow \infty} \partial_x p_0(x) = 0. \end{aligned}$$

DEFINITION 2.1. A pair of functions (p, N) is a classical solution to the Fokker–Planck equation (1.3) in the time interval $[0, T)$ for a given $T \in (0, \infty]$ and with initial data p_0 satisfying (2.1), if the following conditions are satisfied:

1. $N(t)$ is a continuous function for all $t \in [0, T)$.
2. p is continuous in $[0, T) \times [0, \infty)$, $p \in C_t^1 C_x^2((0, T) \times \mathbb{R}^+)$, and for $t \in (0, T)$, $p(t, 0^+)$ is well defined and $p, p_x(t, x) \rightarrow 0$ as $x \rightarrow \infty$.
3. The equation (1.3) is satisfied in the classical sense.

The following result is a simple variant of [11], Theorem 3.1, Theorem 4.2.

LEMMA 2.1 ([11]). *Let p_0 satisfy (2.1). Then there exists a unique classical solution to the Fokker–Planck equation (1.3) in the time interval $[0, T_*)$ for some $T_* > 0$. The maximal time of existence $T_* > 0$ is characterized as*

$$T_* = \sup\{t > 0 : N(t) < \infty\}.$$

The lemma can be proved via a fixed point argument using that

$$\Gamma(N)(t) := 2 \int_0^\infty G(t, s(t) - x) \partial_x p_0(x) dx + \int_0^t N(\tau) G_x(t - \tau, s(t) - s(\tau)) d\tau,$$

which is derived through Green’s identity, defines a contraction mapping Γ on the space of $\{N \in C([0, T]) : \|N\|_\infty \leq M\}$ for some $M > 0$ when T is sufficiently small, see, for example, [11, 20]. Here $s(t) := \int_0^t N(\tau) d\tau$ and

$$(2.2) \quad G(t, x) := \frac{1}{\sqrt{2\pi t}} e^{-|x|^2/2t},$$

is the Green function of the heat equation on the real line. After finding out $N(t)$, $p(t, x)$ can be solved from the first equation in (1.3) on $[0, T]$. The solution can then be extended up to the time of the first blow-up, see Theorem 4.2 [11].

Now we give a proof of Proposition 1.1(1).

PROOF OF PROPOSITION 1.1(1). Suppose by contradiction that a weak solution exists for all time. From the weak formulation in Definition 1.1(1), taking $\phi(t, x) = e^{-\mu x}$ for some $\mu > 0$, we have

$$(2.3) \quad \int_0^\infty e^{-\mu x} p(t, x) dx - \int_0^\infty e^{-\mu x} p_0(x) dx = \int_0^t \int_0^\infty \frac{\mu^2}{2} e^{-\mu x} p(\tau, x) + \alpha \mu e^{-\mu x} N(\tau) p(\tau, x) dx d\tau - \int_0^t N(\tau) d\tau.$$

Since $p \geq 0$, this yields

$$\int_0^\infty e^{-\mu x} p(t, x) dx - \int_0^\infty e^{-\mu x} p_0(x) dx \geq \int_0^t N(\tau) \left(\alpha \mu \int_0^\infty e^{-\mu x} p(\tau, x) dx - 1 \right) d\tau.$$

Writing $M_\mu(t) := \mu \alpha \int_0^\infty e^{-\mu x} p(t, x) dx - 1$, we get

$$M_\mu(t) - M_\mu(0) \geq \mu \alpha \int_0^t N(\tau) M_\mu(\tau) d\tau.$$

By Gronwall’s inequality (see, e.g., [33], Theorem 2.4.5), using $M_\mu(0) = \mu \alpha \int_0^\infty e^{-\mu x} p_0(x) dx - 1 \geq 0$ by (1.6), we have for all $t \geq 0$ that

$$M_\mu(t) = \mu \alpha \int_0^\infty e^{-\mu x} p(t, x) dx - 1 \geq 0.$$

Therefore by (2.3) again, we obtain

$$\int_0^\infty e^{-\mu x} p(t, x) dx - \int_0^\infty e^{-\mu x} p_0(x) dx \geq \frac{\mu^2}{2} \int_0^t \int_0^\infty e^{-\mu x} p(\tau, x) dx ds.$$

This implies that

$$\int_0^\infty e^{-\mu x} p(t, x) dx \geq e^{\frac{\mu^2}{2}t} \int_0^\infty e^{-\mu x} p_0(x) dx \rightarrow \infty \quad \text{as } t \rightarrow \infty,$$

which is impossible. In fact, $e^{\frac{\mu^2}{2}t} \int_0^\infty e^{-\mu x} p_0(x) dx$ cannot be greater or equal than $\int_0^\infty p_0(x) dx$ since otherwise

$$\int_0^\infty p(t, x) dx > \int_0^\infty e^{-\mu x} p(t, x) dx \geq \int_0^\infty p_0(x) dx,$$

but setting $\phi \equiv 1$ (in the domain of $[0, T'] \times [0, \infty)$) in the weak formulation in Definition 1.1(1) reveals that the total mass of p is nonincreasing in time. Solving $e^{\mu^2 T/2} \times \int_0^\infty e^{-\mu x} p_0(x) dx = \int_0^\infty p_0(x) dx$ for T gives the upper bound on the existing time of solutions when (1.6) holds. The second part follows from [21], Theorem 1.1. \square

Super-cooled Stefan problem. Let (p, N) be a classical solution to the Fokker–Planck equation (1.3) in the time interval $[0, T)$. It is well known that the transformation

$$(2.4) \quad u(t, x) := p(t, x - \alpha s(t)), \quad s(t) := \int_0^t N(\tau) d\tau$$

turns the equation into supercooled Stefan problem:

$$(2.5) \quad \begin{cases} u_t = \frac{1}{2} u_{xx} & \text{in } \{(t, x) : x > \alpha s(t), t \in (0, T)\}, \\ s'(t) = \frac{1}{2} u_x(t, \alpha s(t)), \quad u(t, \alpha s(t)) = 0 & \text{for } t \in [0, T), \\ u(0, x) = p_0(x - \alpha s(0)) & \text{for } x \in [\alpha s(0), \infty). \end{cases}$$

We start with the following simple lemma of a comparison principle between two solutions with subquadratic growth in unbounded space-time domains.

LEMMA 2.2. *Let $s_* : [0, T] \rightarrow [0, \infty)$ be a continuous function. Suppose $u_t \leq \frac{1}{2}u_{xx}$ in $D_T := \{(t, x) : x > s_*(t), t \in [0, T)\}$ in the sense of distribution, that is, for any smooth, nonnegative φ that is compactly supported in D_T we have*

$$\int_{D_T} u(t, x) \left[\varphi_t(t, x) + \frac{1}{2} \varphi_{xx}(t, x) \right] dx dt + \int_{s_*(0)}^\infty u_0(x) \varphi(0, x) dx \geq 0.$$

Let $v \geq 0$ satisfy $v_t = \frac{1}{2}v_{xx}$ in D_T . Then if $u(0, \cdot) \leq v(0, \cdot)$ in $(s_*(0), \infty)$,

$$\limsup_{x \rightarrow \infty} \sup_{t \in [0, T)} \left\{ \frac{u(t, x) - v(t, x)}{x^2} \right\} \leq 0 \quad \text{and} \quad \sup_{t \in [0, T)} \limsup_{x \rightarrow s_*(t)} \{u(t, x) - v(t, x)\} \leq 0,$$

we have $u \leq v$ in D_T .

PROOF. For any $\varepsilon > 0$, define

$$v_\varepsilon(t, x) := v(t, x) + \varepsilon t + \varepsilon x^2.$$

By the assumption, for all $M = M(\varepsilon) > 0$ large enough we have $v_\varepsilon(t, M) \geq u(t, M)$ for all $t \in [0, T)$. Since v_ε satisfies the heat equation, and $v_\varepsilon(0, \cdot) \geq u(0, \cdot)$, we can apply the comparison principle (see, e.g., [29], Corollary 6.26) in $\{(t, x) \in D_T : x < M\}$ to conclude that $v_\varepsilon \geq u$ in $\{(t, x) \in D_T : x < M\}$. Passing $M \rightarrow \infty$ yields $v_\varepsilon \geq u$ in D_T . Then passing $\varepsilon \rightarrow 0$ yields $v \geq u$ in D_T . \square

LEMMA 2.3. *Let p_0 be a probability density supported on $(0, \infty)$ that satisfies (1.10). Suppose that a weak solution (p, N) to (1.3) exists for a short time. Then there exists $T_* > 0$ such that (p, N) can be extended to $[0, T_*)$ and (p, N) is a classical solution for $t \in (0, T_*)$. The maximal time of existence $T_* > 0$ is characterized as*

$$T_* = \sup\{t > 0 : N(t) < \infty\}.$$

PROOF. Suppose (p, N) is a weak solution to (1.3) in $[0, T)$ for some $T > 0$. Let $u(t, x) = p(t, x - \alpha s(t))$, and so u is supported in $D_T := \{(t, x) : x > \alpha s(t), t \in [0, T)\}$. Since $0 \leq N(\cdot) \in L^1_{loc}$, $s(t)$ is a continuous, nondecreasing function satisfying $s(0) = 0$. We extend u by 0 to $[0, T) \times \mathbb{R}$. By Definition 1.1(1), we get for any test function $\varphi : [0, T'] \times \mathbb{R} \rightarrow \mathbb{R}$ with $T' < T$, that is smooth and bounded,

$$\begin{aligned} \int_0^\infty u(T', x) \varphi(T', x) dx &= \int_0^{T'} \int_0^\infty u(t, x) \left[\varphi_t(t, x) + \frac{1}{2} \varphi_{xx}(t, x) \right] dx dt \\ &\quad - \int_0^{T'} N(t) \varphi(t, \alpha s(t)) dt + \int_0^\infty u_0(x) \varphi(0, x) dx. \end{aligned}$$

Thus u solves the heat equation in the interior of D_T . Moreover, since $N \geq 0$, we have $u_t \leq \frac{1}{2}u_{xx}$ in $(0, T) \times [0, \infty)$ in the sense of distribution (see Lemma 2.2 with $s_* \equiv 0$).

Since $\sup_{t \in [0, T)} \|u(t, \cdot)\|_{L^1((\alpha s(t), \infty))} < \infty$, by the hypothesis on p ,

$$\eta(t, x) := \int_{\alpha s(t)}^x u(t, y) dy + s(t),$$

is uniformly bounded in D_T . The equation yields that $\eta(t, x)$ solves the heat equation in D_T in the sense of distribution, and η is continuous in both space and time. Then, after restricting η to $(0, T) \times [M, \infty)$ for some $M > \alpha s(T) + 1$, we can obtain an explicit representation

formula for η . Indeed, using the classical reflection method for heat equation with a source on half-line for $\eta(t, x + M) - \eta(t, M)e^{-x}$ (see [35], Section 4.1), we get for all $t \in (0, T)$ and $x > 0$,

$$\begin{aligned}
 &\eta(t, x + M) \\
 &= \eta(t, M)e^{-x} \\
 &\quad + \int_0^\infty (G(t, x - y) - G(t, x + y))(\eta(0, y + M) - \eta(0, M)e^{-y}) dy \\
 &\quad + \int_0^t \int_0^\infty (G(t - \tau, x - y) - G(t - \tau, x + y)) \\
 (2.6) \quad &\quad \times \left(\frac{1}{2} \eta(\tau, M)e^{-y} - \eta_t(\tau, M)e^{-y} \right) dy d\tau \\
 &= \int_0^\infty (G(t, x - y) - G(t, x + y))\eta(0, y + M) dy \\
 &\quad - \int_0^\infty (G_t(t, x - y) - G_t(t, x + y))\eta(0, M)e^{-y} dy \\
 &\quad + \int_0^t \int_0^\infty (G(t - \tau, x - y) - G(t - \tau, x + y)) \frac{1}{2} \eta(\tau, M)e^{-y} dy d\tau,
 \end{aligned}$$

where G is the Green function given by (2.2), and the integral involving η_t is justified by integration by parts. Using the formula (2.6), and $\lim_{x \rightarrow \infty} \eta_x(0, x) = \lim_{x \rightarrow \infty} p_0(x) = 0$ by the assumption, we obtain

$$(2.7) \quad \lim_{x \rightarrow \infty} \sup_{t \in [0, T]} u(t, x) = \lim_{x \rightarrow \infty} \sup_{t \in [0, T]} \eta_x(t, x) = 0.$$

Moreover, it follows from the formula (2.6) (or from the parabolic interior regularity theory [29], Theorem 11.5, and (2.7)) that

$$(2.8) \quad \lim_{x \rightarrow \infty} \partial_x u(t, x) = 0 \quad \text{for all } t \in (0, T).$$

Now set

$$(2.9) \quad \tilde{u}(t, x) := \int_0^\infty (G(t, x - y) - G(t, x + y))p_0(y) dy$$

which is then the solution to the heat equation in $(0, T) \times \mathbb{R}^+$ satisfying $\tilde{u}(t, 0) = 0$ and $\tilde{u}(0, \cdot) = p_0(\cdot)$. The assumption on p_0 yields $\lim_{x \rightarrow \infty} \sup_{t \in [0, T]} \tilde{u}(t, x) = 0$. Due to (2.7), it follows from Lemma 2.2 that $u \leq \tilde{u}$ in $[0, T) \times \mathbb{R}$. Due to the first condition in (1.10), there exists $\delta > 0$ and $h > 0$ such that $\tilde{u} < \frac{1}{\alpha}$ for all $x \in (0, h)$ and $t \in [0, \delta)$. Thus, the same inequality holds for u in place of \tilde{u} . By taking δ to be small, we can assume that $h > s(\delta)$. Then, since u is continuous for $t > 0$, for some M such that $M > s(\delta) + 1$, [19], Theorem 1.3, yields that the following equation possesses a unique classical solution for a short time. For any $\varepsilon \in (0, \delta)$:

$$\begin{cases} \bar{u}_t = \frac{1}{2} \bar{u}_{xx} & \text{in } \{(t, x) : x \in (\alpha \bar{s}(t), M), t > 0\}, \\ \bar{s}'(t) = \frac{1}{2} \bar{u}_x(t, \alpha \bar{s}(t)), \quad \bar{s}(0) = s(\varepsilon) & \text{for } t > 0, \\ \bar{u}(t, \alpha \bar{s}(t)) = 0, \quad \bar{u}(t, M) = g_\varepsilon(t), & \text{for } t > 0, \\ \bar{u}(0, x) = u(\varepsilon, x) & \text{for } x \in (\alpha s(\varepsilon), M), \end{cases}$$

where $g_\varepsilon(t) := u(t + \varepsilon, M)$.

Note that u and \bar{u} are smooth in their support, respectively, for $t > 0$ by parabolic interior Schauder estimates (see [25]). Then applying [18], Theorem 3.1, to

$$\int_{\alpha s(t+\varepsilon)}^x \int_{\alpha \bar{s}(t+\varepsilon)}^y (1 - \alpha u(t + \varepsilon, z)) dz dy \quad \text{and} \quad \int_{\alpha \bar{s}(t)}^x \int_{\alpha \bar{s}(t)}^y (1 - \alpha \bar{u}(t, z)) dz dy,$$

yields $\bar{u}(t, \cdot) = u(t + \varepsilon, \cdot)$ and $\bar{s}(t) = s(t + \varepsilon)$ for all $t \geq 0$ small enough. By (2.7) and (2.8), $p(\frac{\delta}{2}, \cdot) = u(\frac{\delta}{2}, \cdot + \alpha s(\frac{\delta}{2}))$ satisfies (2.1). Finally, Lemma 2.1 yields the conclusion. \square

Self-similar solutions. For any $\beta > 0$, consider the following functions:

$$(2.10) \quad U(t, x; c, \beta) := 2\alpha^{-1}\beta e^{\beta^2} \int_{\beta}^{\frac{x-c}{\sqrt{2t}}} e^{-z^2} dz \quad \text{and} \quad S(t; c, \beta) = \alpha^{-1}(c + \beta\sqrt{2t}).$$

These functions come from self-similar solutions to the supercooled Stefan problem (see [1, 13]): the pair $(U(t, x; c, \beta), S(t; c, \beta))$ satisfies

$$\begin{cases} U_t = \frac{1}{2}U_{xx} & \text{in } \{(t, x) : x > \alpha S(t; c, \beta), 0 < t < T\}, \\ S'(\cdot; c, \beta) = \frac{1}{2}U_x(\cdot, \alpha S(\cdot; c, \beta); c, \beta) & \text{on } (0, \infty). \end{cases}$$

It is easy to see that for all $x, t > 0$,

$$(2.11) \quad \beta_{\infty}(\beta) := \alpha U(0, x; c, \beta) = \lim_{x \rightarrow \infty} \alpha U(t, x; c, \beta) = 2\beta e^{\beta^2} \int_{\beta}^{\infty} e^{-z^2} dz,$$

and for all $\beta > 0$, $\beta_{\infty}(\beta)$ takes all values in $(0, 1)$. Indeed, for all $\beta > 0$,

$$\beta_{\infty}(\beta) = 2\beta \int_0^{\infty} e^{-y^2 - 2\beta y} dy = \int_0^{\infty} e^{-(2\beta)^{-2}z^2 - z} dz < \int_0^{\infty} e^{-z} dz = 1.$$

Moreover, we have the following estimate for all $\beta \geq 1$:

$$(2.12) \quad \beta_{\infty}(\beta) = \int_0^{\infty} e^{-(2\beta)^{-2}z^2 - z} dz \geq \int_0^{\beta} e^{-z} (1 - (2\beta)^{-2}z^2) dz \geq 1 - \frac{1}{\beta^2},$$

where we used that $\int_0^{\infty} e^{-z} z^2 dz = 2$. From the equality in (2.12), we also know that $\beta_{\infty}(\cdot)$ is an increasing function.

2.2. Comparison lemmas. We first present the following comparison lemma. Instead of comparing the solutions to the super-cooled Stefan problem (2.5), we consider a linear combination of the solution and its integration. This will allow us to compare a solution that is possibly large at some points with the self-similar solution U (which is no greater than β_{∞}/α).

LEMMA 2.4. *Suppose $(u_1, s_1), (u_2, s_2)$ are two classical solutions to (2.5) in $[0, T) \times [0, \infty)$. Let $\gamma \in [0, 1]$ and for $i = 1, 2$ write*

$$v_i(t, x) := \gamma u_i(t, x) + \left(\int_{\alpha s_i(t)}^x u_i(t, y) dy + s_i(t) \right).$$

If the following holds for all $t \in [0, T)$:

1. $s_1(t) \leq s_2(t)$,
2. $v_1(0, x) \geq v_2(0, x)$ for all $x > \alpha s_2(0)$,
3. $\liminf_{x \rightarrow \infty} (v_1(t, x) - v_2(t, x)) > 0$ locally uniformly in t ,
4. $v_1(t, \alpha s_2(t)) \geq v_2(t, \alpha s_2(t))$,

then for all $t \in [0, T)$ and $x > \alpha s_2(t)$,

$$v_1(t, x) \geq v_2(t, x).$$

PROOF. Direct computation yields

$$\begin{cases} (v_i)_t = \frac{1}{2}(v_i)_{xx} & \text{in } \{(t, x) : x > \alpha s_i(t), t \in (0, T)\}, \\ v_i(t, \alpha s_i(t)) = s_i(t), (v_i)_x(t, \alpha s_i(t)) = 2\gamma s_i'(t) & \text{for } t \in [0, T). \end{cases}$$

By the assumptions, we have $w := v_1 - v_2$ satisfies

$$w_t = \frac{1}{2}w_{xx} \quad \text{in } \{(t, x) : x > \alpha s_2(t), t \in (0, T)\},$$

and $w(t, \alpha s_2(t)) \geq 0$. Also since the condition on the initial data yields $w(0, \cdot) \geq 0$ on $\{x > \alpha s_2(0)\}$, and due to the condition 3, the conclusion follows from the maximum principle (in bounded space-time domain $\{(t, x) : x \in (\alpha s_2(t), N), t \in [0, T']\}$ for any $T' < T$ and for sufficiently large N) (see, e.g., [29], Corollary 6.26). \square

We also use the following transformation as done in [19]: recall that (u, s) is a solution to (2.5), and define

$$(2.13) \quad m(t, x) := \int_{\alpha s(t)}^x \int_{\alpha s(t)}^y (1 - \alpha u(t, z)) dz dy,$$

and

$$m_0(x) := \int_{\alpha s(0)}^x \int_{\alpha s(0)}^y (1 - \alpha u(0, z)) dz dy.$$

Then m satisfies the following problem:

$$(2.14) \quad \begin{cases} m_t = \frac{1}{2}m_{xx} - 1 & \text{in } \{(t, x) : x > \alpha s(t), 0 < t < T\}, \\ m(t, \alpha s(t)) = m_x(t, \alpha s(t)) = 0 & \text{for } t \in [0, T), \\ m(0, x) = m_0(x) & \text{for } x \in [\alpha s(0), \infty). \end{cases}$$

Note that in (2.14), differentiation of $s(t)$ is not involved. However intuitively $s(t)$ can still be identified through the equation because of the two boundary information (if known $s(t)$, only one boundary data is needed to solve for m). We have the following comparison principle.

LEMMA 2.5. *Suppose $(u_1, s_1), (u_2, s_2)$ are two classical solutions to (2.5) in $[0, T) \times [0, \infty)$ with nonnegative initial data $u_{1,0}, u_{2,0}$ that are supported in $(\alpha s_1(0), \infty)$ and $(\alpha s_2(0), \infty)$ respectively. Write their corresponding transformations as m_1, m_2 . Suppose the following holds for all $t \in [0, T)$:*

1. $(s_2(0) \leq s_1(0), s_2'(0) < s_1'(0))$ or $(s_2(0) < s_1(0))$,
2. $m_2(0, \cdot) \geq m_1(0, \cdot)$ in $[0, \infty)$,
3. $m_2(t, x) \geq 0$ for $x > \alpha s_2(0)$,
4. $\liminf_{x \rightarrow \infty} (m_2(t, x) - m_1(t, x)) > 0, \liminf_{x \rightarrow \infty} m_2(t, x) > 0$.

Then $s_2(t) < s_1(t)$, and $m_1(t, \cdot) < m_2(t, \cdot)$ for all $t \in (0, T)$.

PROOF. The proof is identical to that in Lemma 3.1 and Remark 3.2 in [19]. \square

Recall the self-similar solutions (U, S) given in (2.10). The idea of controlling $N(t)$ in (1.3) and then proving long time existence of solution is to apply the above two comparison principles to compare the free boundaries of a general solution u and the self-similar solution U with certain choices of c, β . We first show that $S(t; c, \beta) \geq s(t)$ for $c > 0$.

LEMMA 2.6. *Let (u, s) be a classical solution to the super-cooled Stefan problem (2.5) for $t \in [0, T)$, such that*

$$(2.15) \quad \|u_0\|_{L^1(\mathbb{R}^+)} \leq 1, \quad \|u_0\|_\infty < \infty, \quad \lim_{x \rightarrow 0} u_0(x) + \lim_{x \rightarrow \infty} u_0(x) = 0,$$

and

$$(2.16) \quad \int_0^x (1 - \alpha u_0(y)) dy > 0 \quad \text{for all } x \in (0, \infty).$$

There exists $C_0 \geq 2$ depending only on u_0 such that for all $\beta \geq C_0$, we have

$$s(t) \leq S(t; 0, \beta) = \alpha^{-1} \beta \sqrt{2t} \quad \text{for all } t \in [0, T).$$

PROOF. Recall that $u(t, x)$ solves the supercooled Stefan problem:

$$u_t = \frac{1}{2} u_{xx} \quad \text{in } \{x \in (\alpha s(t), \infty), t \in [0, T)\},$$

$$u(t, \alpha s(t)) = 0 \quad \text{and} \quad u_x(t, \alpha s(t)) = s'(t).$$

Since $s(\cdot) \geq 0$, comparison principle yields $u \leq \tilde{u}$ where the latter is given by (2.9). By the assumption $u(0, x) \rightarrow 0$ as $x \rightarrow \infty$, we have $\lim_{x \rightarrow \infty} \tilde{u}(t, x) = 0$ (uniformly in t), which implies that $\lim_{x \rightarrow \infty} u(t, x) = 0$ uniformly in t . Thus for all $t \in [0, T)$ we have

$$(2.17) \quad \liminf_{x \rightarrow \infty} U(t, x; c, \beta) = \alpha^{-1} \beta_\infty > 0 = \lim_{x \rightarrow \infty} u(t, x).$$

Let m be defined as in (2.13), and we also define

$$M(t, x; c, \beta) := \int_{\alpha S(t)}^x \int_{\alpha S(t)}^y (1 - \alpha U(t, z; c, \beta)) dz dy.$$

It follows from (2.17) that

$$(2.18) \quad \liminf_{x \rightarrow \infty} (m(t, x) - M(t, x; c, \beta)) > 0.$$

Since the total mass of $u(t, \cdot)$ is bounded from above by 1, we know

$$(2.19) \quad m_0(x) = \int_0^x \int_0^y (1 - \alpha u_0(z)) dz dy \geq \frac{x^2}{2} - \alpha x.$$

Also by the assumption (2.15)–(2.16), we obtain $m_0(x) > 0$ for all $x \in (0, \infty)$ and $m_0(x) \geq \frac{x^2}{4}$ for $x > 0$ small enough. In view of (2.11) and (2.12), there exists $C_0 \geq 2$ depending only on u_0 such that for all $c \geq 0$, if $\beta \geq C_0$, we have

$$M(0, \cdot; c, \beta) = \frac{1 - \beta_\infty}{2} (x - c)_+^2 \leq \frac{x^2}{2\beta^2} < m_0(\cdot) \quad \text{in } (0, 3\alpha].$$

While for $x > 3\alpha$, (2.19) yields

$$M(0, \cdot; c, \beta) \leq \frac{x^2}{8} < \frac{x^2}{2} - \alpha x \leq m_0(\cdot).$$

Note that $S(0; c, \beta) = \alpha^{-1}c$. Then it follows from Lemma 2.5 with $u_1 = U, u_2 = u$ that $s(t) < S(t; c, \beta)$ for all $t \in [0, T)$ and $c > 0$. By passing $c \rightarrow 0$, we get

$$(2.20) \quad s(t) \leq S(t; c, \beta) \quad \text{for all } t \in [0, T) \text{ and } c \geq 0. \quad \square$$

In Lemma 2.6, we only applied the second comparison lemma (Lemma 2.5). From its proof, note that we can replace the assumption (2.16) by $\int_0^x \int_0^y (1 - \alpha u_0(z)) dz dy > 0$ for all $x > 0$, which is slightly weaker than (2.16).

In order to compare $s(t), S(t; c, \beta)$ for $c < 0$, we also need the following computations. For some $\gamma \in (0, 1)$ to be determined, write

$$(2.21) \quad V(t, x; c, \beta) := \gamma U(t, x; c, \beta) + \left(\int_{\alpha S(t; c, \beta)}^x U(t, y; c, \beta) dy + S(t; c, \beta) \right)$$

and

$$(2.22) \quad v(t, x) := \gamma u(t, x) + \left(\int_{\alpha s(t)}^x u(t, y) dy + s(t) \right).$$

LEMMA 2.7. *Under the assumptions of Lemma 2.6, there exists $\gamma_0 \in (0, 1)$ (depending only on u_0) such that for all $\gamma \in (0, \gamma_0)$, and β satisfying*

$$\beta \geq \max\{C_0, 100, 2\sqrt{\alpha/\gamma}, 4\sqrt{2T}\gamma^{-1}, 4\sqrt{2T}(\alpha\gamma)^{-1}, 4\sqrt{2T}\gamma^{-1}(\ln(4\sqrt{2T}\gamma^{-1}))^2\},$$

and $c \in (-\alpha^{-1}\beta\sqrt{2T}, 0)$, if $s(t) \geq S(t; c, \beta)$ for all $t \in [0, t_1]$ for some $t_1 < T$, then the following inequalities hold:

$$(2.23) \quad \liminf_{x \rightarrow \infty} (V(t, x; c, \beta) - v(t, x)) = \infty \quad \text{for all } t \in [0, T),$$

$$(2.24) \quad V(0, \cdot; c, \beta) \geq v(0, \cdot) \quad \text{in } [0, \infty),$$

$$(2.25) \quad V(t, x; c, \beta) \geq \alpha^{-1}x \quad \text{for all } t \in [0, t_1] \text{ and } x \leq \beta\sqrt{2T}.$$

PROOF. Due to (2.17), clearly (2.23) holds for all t, c, β . Using (2.11), (2.12), and $c \in (-\alpha^{-1}\beta\sqrt{2T}, 0)$ yields for all $x \geq 0$,

$$V(0, x; c, \beta) \geq \gamma\alpha^{-1}\beta_\infty + \alpha^{-1}\beta_\infty x - (\alpha\beta)^{-1}\sqrt{2T}.$$

Let us assume $\beta \geq \max\{4\gamma^{-1}\sqrt{2T}, 2\}$ and then $\beta_\infty \geq \frac{3}{4}$. By (2.12) and (2.21), we get

$$(2.26) \quad \begin{aligned} &V(0, x; c, \beta) - v(0, x) \\ &\geq \gamma \left(\frac{3}{4}\alpha^{-1} - u(0, x) \right) - \gamma(4\alpha)^{-1} - \alpha^{-1}(1 - \beta_\infty)x + \alpha^{-1} \int_0^x (1 - \alpha u_0(y)) dy \\ &\geq \gamma((2\alpha)^{-1} - u(0, x)) + \alpha^{-1} \left(-\beta^{-2}x + \int_0^x (1 - \alpha u_0(y)) dy \right). \end{aligned}$$

In view of (2.15), there exists $c, A > 0$ such that

$$u(0, x) \leq (4\alpha)^{-1} \quad \text{for all } x \in [0, c] \quad \text{and} \quad u(0, x) \leq A \quad \text{for all } x \in [0, \infty).$$

When $x \geq 2\alpha$, the right-hand side of (2.26)

$$\geq -\gamma A + \alpha^{-1}(-\beta^{-2}x + x - \alpha) \geq -\gamma A + (1 - 2\beta^{-2}) \geq 0,$$

if $\beta \geq 2$ and $\gamma \leq (2A)^{-1}$. Next when $x \leq c$ (then $u(0, x) \leq (4\alpha)^{-1}$), (2.26) and $\beta \geq 2$ yield again $V(0, x; c, \beta) - v(0, x) \geq 0$. Lastly by the assumption (2.16), there exists $\varepsilon > 0$ such that $\int_0^x (1 - \alpha u_0(y)) dy \geq \varepsilon$ for $x \in [c, 2\alpha]$. Thus we get

$$V(0, x; c, \beta) - v(0, x) \geq \gamma((2\alpha)^{-1} - A) + (\alpha^{-1}\varepsilon - 2\beta^{-2}) \geq 0$$

if $\beta \geq 2\sqrt{\alpha/\varepsilon}$ and $\gamma = \gamma(\varepsilon, A) \leq \varepsilon$ is small enough. Overall, we find that there exists γ depending only on u_0 such that (2.24) holds for all $\beta \geq \max\{4\gamma^{-1}\sqrt{2T}, 2\sqrt{\alpha/\gamma}, 2\}$.

To prove the last inequality (2.25), we need a lower bound on U . Below we write $S(t) := S(t; c, \beta)$ for abbreviation of notation. It follows from (2.10), (2.12), and the fact $\int_0^\infty e^{-z} z^2 dz = 2$ that for $\beta \geq 2$, if $\frac{x-\alpha S(t)}{\sqrt{2t}} \geq \frac{2\ln\beta}{\beta}$, we have

$$\begin{aligned} \alpha U(t, x; c, \beta) &= 2\beta e^{\beta^2} \int_{\beta}^{\frac{x-c}{\sqrt{2t}}} e^{-(z')^2} dz' = \int_0^{2\beta(\frac{x-\alpha S(t)}{\sqrt{2t}})} e^{-z-(2\beta)^{-2}z^2} dz \\ (2.27) \quad &\geq \int_0^{2\beta(\frac{x-\alpha S(t)}{\sqrt{2t}})} e^{-z}(1-(2\beta)^{-2}z^2) dz \geq \int_0^{2\beta(\frac{x-\alpha S(t)}{\sqrt{2t}})} e^{-z} dz - \frac{1}{2\beta^2} \\ &= 1 - \exp\left(-2\beta\left(\frac{x-\alpha S(t)}{\sqrt{2t}}\right)\right) - \frac{1}{2\beta^2} \geq 1 - \frac{1}{\beta^2}. \end{aligned}$$

When $\beta \geq 2$ and $\frac{x-\alpha S(t)}{\sqrt{2t}} < \frac{2\ln\beta}{\beta}$, since $\frac{1}{8} \geq \frac{\ln\beta}{2\beta^3}$, direct computation yields

$$\begin{aligned} \alpha U(t, x; c, \beta) &= \int_0^{2\beta(\frac{x-\alpha S(t)}{\sqrt{2t}})} e^{-z-(2\beta)^{-2}z^2} dz \geq \int_0^{2\beta(\frac{x-\alpha S(t)}{\sqrt{2t}})} e^{-z-(2^{-1}\beta^{-3}\ln\beta)z} dz \\ (2.28) \quad &\geq \frac{8}{9} \left(1 - \exp\left(-\frac{9\beta}{4}\left(\frac{x-\alpha S(t)}{\sqrt{2t}}\right)\right)\right) \\ &\geq \frac{1}{2} \min\left\{\beta\left(\frac{x-\alpha S(t)}{\sqrt{2t}}\right), 1\right\}. \end{aligned}$$

Using these estimates, for any $t \in [0, T)$ and

$$x \in \left[\alpha S(t), \frac{2\sqrt{2t} \ln \beta}{\beta} + \alpha S(t)\right),$$

we obtain

$$V(t, x; c, \beta) \geq \frac{\gamma}{2\alpha} \min\left\{\beta\left(\frac{x-\alpha S(t)}{\sqrt{2t}}\right), 1\right\} + S(t)$$

and so to have $V(t, x; c, \beta) \geq \alpha^{-1}x$, it suffices to require $\beta \geq 2\gamma^{-1}\sqrt{2T}$ and $\frac{\beta}{\ln\beta} \geq 4\gamma^{-1}\sqrt{2T}$ which is indeed guaranteed by the assumption on β . Next for

$$(2.29) \quad x \in \left[\frac{2\sqrt{2t} \ln \beta}{\beta} + \alpha S(t), \beta\sqrt{2T}\right],$$

by (2.27) and (2.28) we find (writing $U(t, y) := U(t, y; c, \beta)$)

$$\begin{aligned} &\alpha \int_{\alpha S(t)}^x U(t, y; c, \beta) dy + \alpha S(t) \\ &= \alpha \int_{\alpha S(t)}^{\alpha S(t) + \frac{\sqrt{2t}}{\beta}} U(t, y) dy \\ &\quad + \alpha \int_{\alpha S(t) + \frac{\sqrt{2t}}{\beta}}^{\alpha S(t) + \frac{2\sqrt{2t} \ln \beta}{\beta}} U(t, y) dy + \alpha \int_{\alpha S(t) + \frac{2\sqrt{2t} \ln \beta}{\beta}}^x U(t, y) dy + \alpha S(t) \\ &\geq \sqrt{2t} \left(\frac{1}{4\beta} + \frac{1}{2} \left(\frac{2\ln\beta}{\beta} - \frac{1}{\beta}\right) + \left(1 - \frac{1}{\beta^2}\right) \left(\frac{x-\alpha S(t)}{\sqrt{2t}} - \frac{2\ln\beta}{\beta}\right)\right) + \alpha S(t) \\ &\geq -\frac{\sqrt{2t}}{4\beta} - \frac{\sqrt{2t} \ln \beta}{\beta} + \left(1 - \frac{1}{\beta^2}\right)x + \frac{\alpha S(t)}{\beta^2}. \end{aligned}$$

Due to $\alpha S(t) \geq c \geq -\alpha^{-1}\beta\sqrt{2T}$, then for x satisfying (2.29) we obtain

$$\begin{aligned} \alpha V(t, x; c, \beta) - x &\geq \gamma \left(1 - \frac{1}{\beta^2}\right) - \frac{\sqrt{2t}}{4\beta} - \frac{\sqrt{2t} \ln \beta}{\beta} - \frac{x}{\beta^2} + \frac{\alpha S(t)}{\beta^2} \\ &\geq \frac{3}{4}\gamma - \sqrt{2T} \left(\frac{5}{4\beta} + \frac{\ln \beta}{\beta} + \frac{1}{\alpha\beta}\right) \geq 0, \end{aligned}$$

whenever

$$\beta \geq \max\{100, 4\sqrt{2T}(\alpha\gamma)^{-1}, 4\sqrt{2T}\gamma^{-1}(\ln(4\sqrt{2T}\gamma^{-1}))^2\}.$$

This is because for $c_0 := 4\sqrt{2T}\gamma^{-1} > 0$, either $\frac{\beta}{\ln \beta} \geq \frac{100}{\ln 100} \geq c_0$ or $\frac{\beta}{\ln \beta} \geq \frac{c_0(\ln c_0)^2}{\ln(c_0(\ln c_0)^2)} \geq c_0$. We proved (2.25). \square

In the following proposition we show that if the curves $x = \alpha s(t)$ and $x = \alpha S(t; c, \beta)$ with $c < 0$ intersect at time $t = t_0 > 0$, then they can only insect at $t = t_0$ for all $t \in [0, T)$.

LEMMA 2.8. *Under the assumptions of Lemma 2.6, let (γ, β) be from Lemma 2.7. For any fixed $t_0 \in (0, T)$, if there is a value $c \in (-\alpha^{-1}\beta\sqrt{2T}, 0)$ such that $s(t_0) = S(t_0; c, \beta)$, then for all $t \in (0, T)$,*

(2.30) $s(t) - S(t; c, \beta)$ changes sign from positive to negative at $t = t_0$
(i.e., $s(t) - S(t; c, \beta) > 0$ for all $t < t_0$ and $s(t) - S(t; c, \beta) < 0$ for all $t > t_0$).

PROOF. It follows from $c < 0$ and $S(0; c, \beta) = \alpha^{-1}c < 0$ that $S(t; c, \beta) < s(t)$ for t sufficiently small. Suppose for contradiction that there is $t_1 < t_0$ such that $S(t_1; c, \beta) = s(t_1)$ and $S(t; c, \beta) < s(t)$ for $t < t_1$.

Lemma 2.6 yields $s(t) \leq \alpha^{-1}\beta\sqrt{2T}$ for all $t \in [0, T)$. Hence it follows from (2.25) that

$$V(t, \alpha s(t); c, \beta) \geq s(t) = v(t, \alpha s(t)) \quad \text{for } t \in [0, T).$$

Then, using the assumption that $S(t; c, \beta) < s(t)$ for $t < t_1$, Lemma 2.7 and Lemma 2.4 (with $v_1 = V, v_2 = v$ where V, v are given in (2.21), (2.22) respectively) yield that

(2.31)
$$\begin{aligned} \gamma U(t, x; c, \beta) + \int_{\alpha S(t)}^x U(t, y; c, \beta) dy + S(t; c, \beta) &= V(t, x; c, \beta) \\ &\geq v(t, x) = \gamma u(t, x) + \int_{\alpha s(t)}^x u(t, y) dy + s(t) \end{aligned}$$

for all $(t, x) \in \{x > \alpha s(t), t \in [0, t_1]\}$. By the strong maximum principle (or Hopf’s lemma), we have $S'(t_1; c, \beta) > s'(t_1)$.

Next consider

$$Z(x) := \int_{\alpha s(t_1)}^x \int_{\alpha s(t_1)}^y (U(t_1, z; c, \beta) - u(t_1, z)) dz$$

which, by (2.31) and the assumption that $S(t_1; c, \beta) = s(t_1)$, satisfies

$$\gamma Z''(x) + Z'(x) \geq 0 \quad \text{and} \quad Z(\alpha s(t_1)) = Z'(\alpha s(t_1)) = 0.$$

This implies that $Z(x) \geq 0$ for all $x > \alpha s(t_1)$. Therefore the definitions of m, M yield $m(t_1, \cdot) \geq M(t_1, \cdot; c, \beta)$. Clearly for all $t \in [0, T)$, by maximum principle and (2.18), we have that $m(t, \cdot) > 0$ for $x > \alpha s(t)$. In view of Lemma 2.5 again, we obtain $S(t; c, \beta) > s(t)$ for all $t > t_1$ which contradicts with the assumption that $S(t_0; c, \beta) = s(t_0)$ with $t_0 > t_1$. Hence $S(t; c, \beta) < s(t)$ for all $t < t_0$. By going over the arguments in the above again, we also find that $S(t; c, \beta) > s(t)$ for all $t > t_0$. \square

2.3. *Proofs of Theorem 1.1 and Corollary 1.2.* We start by proving Theorem 1.1.

PROOF OF THEOREM 1.1. Suppose (p, N) is a weak solution to (1.3) in $[0, T]$. Let $s(t) = \int_0^t N(r) dr$ for all $t \in [0, T]$, and u, m be defined as in (2.4), (2.13) respectively. Then $u_0 = p_0$. Since u solves the heat equation, $u(t, \cdot)$ is bounded in L^∞ norm for any $t \in (0, T)$ (see, e.g., [29], Theorem 6.17). By the argument before (2.17), we know that $u(t, x) \rightarrow 0$ as $x \rightarrow \infty$ locally uniformly in t and so the same property holds for $p(t, x)$. Therefore there exists $C > 0, \varepsilon \in (0, T)$ such that

$$\int_0^x (1 - \alpha p(t, y)) dy > 0 \quad \text{for all } x \geq C \text{ and } t \leq \varepsilon.$$

Due to (1.10) and $u \leq \tilde{u}$ with \tilde{u} from Lemma 2.3, we can assume without loss of generality that for the same $C, \varepsilon > 0, p(t, x) < \frac{1}{\alpha}$ for all $t \leq \varepsilon$ and $x \leq \frac{1}{C}$. By further taking ε to be small enough, the weak formulation of solutions and the assumption on p_0 imply that $\int_0^x (1 - \alpha p(t, y)) dy > 0$ for $x \in [\frac{1}{C}, C]$ and $t \leq \varepsilon$. Then the assumptions (2.15)–(2.16) hold with u_0 replaced by $u(\varepsilon, \cdot)$. Hence, by starting at a small time $t = \varepsilon$ instead of $t = 0$, we can assume without loss of generality that p_0 is uniformly bounded in L^∞ , and the solution (p, N) is a classical solution.

We take $\beta > 0$ to be the smallest constant satisfying the condition in Lemma 2.7. For any fixed $t_0 \in (0, T)$, (2.20) implies $s(t_0) \leq S(t_0; 0, \beta)$. First, if $s(t_0) = S(t_0; 0, \beta)$, we claim that $s(t) = S(t; 0, \beta)$ for all $t \leq t_0$. If this is not true, then there exist $t_1 < t_0$ and $c < 0$ sufficiently close to 0 such that $s(t_1) < S(t_1; c, \beta)$. According to Lemma 2.8, we must have $s(t) < S(t; c, \beta)$ for all $t > t_1$ which is a contradiction because then $s(t_0) < S(t_0; c, \beta) < S(t_0; 0, \beta)$. So in this case we obtain $s(t) = S(t; 0, \beta)$ for all $t \leq t_0$. Also, in the case, for all $t \in (0, t_0)$ we have

$$N(t) = s'(t) = S'(t; 0, \beta) = \alpha^{-1} \beta (2t)^{-\frac{1}{2}}.$$

Next we consider the case when $s(t_0) < S(t_0; 0, \beta)$, which by definition is the same as $s(t_0) < \alpha^{-1} \beta \sqrt{2t_0}$. Thus Lemma 2.8 yields that the curve $x = S(t; s(t_0) - \alpha^{-1} \beta \sqrt{2t_0}, \beta)$ intersects with $x = s(t)$ at exactly one point $t = t_0$ for all $t \in [0, T]$ (notice here in terms of $S(t; c, \beta)$, c takes the value of $s(t_0) - \alpha^{-1} \beta \sqrt{2t_0} \geq -\alpha^{-1} \beta \sqrt{2T}$, and so the assumption of Lemma 2.8 is satisfied). Therefore

$$N(t) = s'(t) \leq S'(t; s(t) - \alpha^{-1} \beta \sqrt{2t}, \beta) = \alpha^{-1} \beta (2t)^{-\frac{1}{2}}.$$

From the choice of β , and by varying T (to be t) in the above arguments, we obtain for all $t \in [0, T)$,

$$N(t) \leq \alpha^{-1} \beta (2t)^{-\frac{1}{2}} \leq C \alpha^{-1} (1 + \alpha^{-1} + (1 + \alpha^{\frac{1}{2}}) t^{-\frac{1}{2}} + (\ln t)^2)$$

for some C depending only on p_0 . We can now conclude the proof by Lemma 2.3. \square

Now we proceed to proving Corollary 1.2.

PROOF OF COROLLARY 1.2. To prove the first statement, in view of Theorem 1.1, it suffices to have

$$(2.32) \quad \int_0^x (1 - \alpha \delta_{x_0}(y)) dy > 0$$

for all $x > 0$. Direct computation yields that this is equivalent to $x > \alpha 1_{x > x_0}$, which is the same as $\alpha < x_0$.

For the second statement, it follows from Proposition 1.1 that if

$$\alpha > 2 \int_0^\infty x \delta_{x_0} dx = 2x_0,$$

then any weak solution cannot exist for all time. In view of Lemma 2.3, $(p_{x_0})_x \rightarrow \infty$ in finite time. \square

3. Solutions to the Fokker–Planck equation (1.5). In this section, we consider the Fokker–Planck equation (1.5). First of all, comparing to the equation (1.3), the equation is nonlocal. Since the total mass is decreasing after assuming $\lim_{x \rightarrow \infty} q(t, x) = 0$ for all t (there is no mass coming from $x = \infty$), the nonlocal form of $\lambda(t)$ yields a fast growth of it as t increases. Hence it is more likely that $\lambda(t)$ grows to infinity in finite time comparing to $N(t)$ in (1.3).

Recall the following result from [32], Proposition 4.1, which proves existence and uniqueness of a generalized solution up to the first blow-up.

LEMMA 3.1 ([32]). *Let $q_0(\cdot) \in W_2^1([0, \infty))$ with $q_0(0) = 0$. For any $T > 0$, there exist a time $t_{\text{reg}} \in (0, T]$ and a function $\lambda \in L_{\text{loc}}^2([0, t_{\text{reg}}))$ such that for all $T' \in (0, t_{\text{reg}})$ the unique solution to the Fokker–Planck equation (1.5) in $W_2^{1,2}([0, T'] \times [0, \infty))$ satisfies*

$$\lambda(t) = -\frac{1}{2} \frac{q_x(t, 0)}{\int_0^\infty q(t, y) dy} \quad \text{for almost every } t \in [0, T']$$

and $\lim_{T' \uparrow t_{\text{reg}}} \|\lambda\|_{L^2[0, T']} = \infty$ if $t_{\text{reg}} < T$.

The idea is to prove $t_{\text{reg}} = \infty$ for suitable q_0, β , and α . Let us perform a transformation which turn the nonlocal equation (1.5) into a local one. Denoting $\bar{q}(t) := \int_0^\infty q(t, x) dx$, we then have $\bar{q}'(t) = \lambda(t)\bar{q}(t)$, and $r := q/\bar{q}$ satisfies

$$(3.1) \quad \begin{cases} r_t = \frac{1}{2} r_{xx} - (\alpha\lambda(t) + \beta)r_x - \lambda(t)r & \text{in } [0, \infty)^2, \\ \lambda(t) = -\frac{1}{2} r_x(t, 0), \quad r(t, 0) = 0 & \text{on } [0, \infty), \\ r(0, x) = q_0(x)/\bar{q}(0, \cdot) & \text{on } [0, \infty). \end{cases}$$

Clearly the equation preserves mass. Now let us prove Proposition 1.1(2). Call $r_0(x) := q_0(x)/\bar{q}(0, \cdot)$.

PROOF OF PROPOSITION 1.1(2). Let (r, λ) be from (3.1) and suppose the solution exists for $t \in [0, T']$. Since $q \in W_2^{1,2}([0, T'] \times \mathbb{R})$, then $r \in W_2^{1,2}([0, T'] \times \mathbb{R})$ and $\int_0^\infty e^{-\mu x} r(t, x) dx$ is Hölder continuous in time by Morrey’s inequality. It follows from the equation that

$$(3.2) \quad \begin{aligned} \int_0^\infty e^{-\mu x} r(t, x) dx &= \int_0^\infty e^{-\mu x} r_0(x) dx + \int_0^t \int_0^\infty (2^{-1} e^{-\mu x} r_{xx}(\tau, x) \\ &\quad - (\alpha\lambda(\tau) + \beta)e^{-\mu x} r_x(\tau, x) - \lambda(\tau)e^{-\mu x} r(\tau, x)) dx d\tau \\ &= \int_0^\infty e^{-\mu x} r_0(x) dx + (2^{-1}\mu^2 - \mu\beta) \int_0^t \int_0^\infty e^{-\mu x} r(\tau, x) dx d\tau \\ &\quad - \int_0^t \lambda(\tau) \left((1 + \alpha\mu) \int_0^\infty e^{-\mu x} r(\tau, x) dx - 1 \right) d\tau. \end{aligned}$$

Since $\mu > 2\beta$, writing $M_\mu(t) := (1 + \alpha\mu) \int_0^\infty e^{-\mu x} r(t, x) dx - 1$, we get

$$M_\mu(t) \geq M_\mu(0) - (1 + \alpha\mu) \int_0^t \lambda(\tau) M_\mu(\tau) d\tau.$$

It then follows from Gronwall’s inequality and $M_\mu(0) \geq 0$ by (1.8) that for all $t \geq 0$,

$$(1 + \alpha\mu) \int_0^\infty e^{-\mu x} r(t, x) dx \geq 1.$$

Hence (3.2) yields

$$\int_0^\infty e^{-\mu x} r(t, x) dx \geq \int_0^\infty e^{-\mu x} r_0(x) dx + \frac{1}{2} \mu(\mu - 2\beta) \int_0^t \int_0^\infty e^{-\mu x} r(\tau, x) dx d\tau,$$

which, by Gronwall’s inequality, implies

$$\int_0^\infty e^{-\mu x} r(t, x) dx \geq e^{\frac{\mu(\mu-2\beta)}{2}t} \int_0^\infty e^{-\mu x} r_0(x) dx.$$

However since the total mass of $r(t, \cdot)$ is always 1. The solution cannot exist for $t \geq T$ where T is such that

$$e^{\frac{\mu(\mu-2\beta)}{2}T} \int_0^\infty e^{-\mu x} r_0(x) dx = 1.$$

Then Lemma 3.1 yields that the L^2 norm of $\lambda(t)$ blows up at the time when the solution fails to exist. \square

To prove Theorem 1.2, we show an L^2 bound on $\lambda(t)$ for general initial data with small α and large $\beta > 0$. We need the following technical lemma which is similar to [12], Lemma 3.4.

LEMMA 3.2. Define $\phi : [0, \infty) \rightarrow \mathbb{R}$ by $\phi(x) = e^{-\frac{1}{1-x^2}}$ if $x \in [0, 1)$, and $\phi(x) := 0$ otherwise. Then the following properties hold:

1. $0 \leq -\phi_x \leq \phi$ on $(0, \frac{1}{4})$.
2. There exists $C > 0$ such that $\phi_x^2 + \phi_{xx}^2 + \phi_{xxx}^2 \leq C\phi$ on $(0, 1)$.
3. There exists $x_0 \in (0, 1)$ such that $\phi_{xx}(x_0) = 0$ and $\phi_{xx} \leq 0$ on $(0, x_0)$.

PROOF. These are results of trivial (but tedious) computations. For the first property, direct computation yields for $x \in (0, \frac{1}{4})$, $-\phi_x = \frac{2x}{(1-x^2)^2} \phi \leq \phi$. For the remaining claims, it follows line by line from the proof of [12], Lemma 3.4. \square

PROOF OF THEOREM 1.2. For any fixed $\kappa \in (0, \frac{1}{8}]$, define $\omega(x) := 2\kappa x e^{-\sqrt{2\kappa}x}$ and then it is not hard to check that $\omega_x(0) = 2\kappa$ and

$$(3.3) \quad \frac{1}{2} \omega_{xx} + \sqrt{2\kappa} \omega_x + \kappa \omega = 0.$$

Since $\kappa \leq \frac{1}{8}$, $\omega(\cdot)$ is an increasing function in $(0, \frac{1}{\sqrt{2\kappa}}) \supseteq (0, 1)$. Note that $\omega(\cdot)$ can be viewed as a stationary solution to (3.1) with $\lambda(t) \equiv -\kappa$ and any α, β satisfying $\alpha\kappa - \beta = \sqrt{2\kappa}$.

Recall $r = \frac{q}{\alpha}$ and we call

$$(3.4) \quad h(t, x) := \frac{r(t, x)}{\omega(x)} \quad \text{and} \quad j(t) := -\frac{\lambda(t)}{\kappa}.$$

By L'Hopital's rule, we have $h(t, 0) = j(t)$. Then let $\phi(x)$ be from Lemma 3.2 and recall that it is supported on $(0, 1)$. Below we will sometimes drop t, x from the notation of $h(t, x), r(t, x), \omega(x), j(t), \lambda(t), \phi(x)$. It follows from equation (3.1) that

$$\begin{aligned} \frac{d}{dt} \int_0^\infty h^2 \omega \phi \, dx &= \int_0^\infty h(r_{xx} - 2(\alpha\lambda + \beta)r_x - 2\lambda r)\phi \, dx \\ &= \int_0^\infty (-h_x r_x - 2(\alpha\lambda + \beta)r_x - 2\lambda r)\phi \, dx - \int_0^\infty h r_x \phi_x \, dx + 2\lambda j\phi(0). \end{aligned}$$

Using that $r_x = h_x \omega + h \omega_x$ and $\phi(0) = e^{-1}$, we get the above

$$\begin{aligned} &= - \int_0^\infty (h_x^2 \omega + h h_x \omega_x)\phi \, dx - 2(\alpha\lambda + \beta) \int_0^\infty h(h_x \omega + h \omega_x)\phi \, dx \\ &\quad - 2\lambda \int_0^\infty h^2 \omega \phi \, dx - \int_0^\infty h r_x \phi_x \, dx - 2e^{-1} j^2 \kappa \\ &= - \int_0^\infty h_x^2 \omega \phi \, dx + \frac{1}{2} \int_0^\infty h^2 \omega_{xx} \phi \, dx + \frac{1}{2} \int_0^\infty h^2 \omega_x \phi_x \, dx - (\alpha\lambda + \beta) \int_0^\infty h^2 \omega_x \phi \, dx \\ &\quad - 2\lambda \int_0^\infty h^2 \omega \phi \, dx + (\alpha\lambda + \beta) \int_0^\infty h^2 \omega \phi_x \, dx - \int_0^\infty h(h_x \omega + h \omega_x)\phi_x \, dx - e^{-1} j^2 \kappa \\ &= - \int_0^\infty h_x^2 \omega \phi \, dx + \frac{1}{2} \int_0^\infty h^2 \omega_{xx} \phi \, dx + \frac{1}{2} \int_0^\infty h^2 \omega \phi_{xx} \, dx - (\alpha\lambda + \beta) \int_0^\infty h^2 \omega_x \phi \, dx \\ &\quad - 2\lambda \int_0^\infty h^2 \omega \phi \, dx + (\alpha\lambda + \beta) \int_0^\infty h^2 \omega \phi_x \, dx - \frac{\lambda^2}{e\kappa}. \end{aligned}$$

Using equation (3.3), we obtain

$$\begin{aligned} \frac{d}{dt} \int_0^\infty h^2 \omega \phi \, dx &= - \int_0^\infty h_x^2 \omega \phi \, dx - (\alpha\lambda + \beta + \sqrt{2\kappa}) \int_0^\infty h^2 \omega_x \phi \, dx \\ (3.5) \quad &\quad - (2\lambda + \kappa) \int_0^\infty h^2 \omega \phi \, dx + (\alpha\lambda + \beta) \int_0^\infty h^2 \omega \phi_x \, dx \\ &\quad + \frac{1}{2} \int_0^\infty h^2 \omega \phi_{xx} \, dx - \frac{\lambda^2}{e\kappa}. \end{aligned}$$

Let us write $I(t) := \int_0^\infty h^2 \omega \phi \, dx$, and $J(t) := \int_0^\infty h_x^2 \omega \phi \, dx$. By Young's inequality, we get

$$-\alpha\lambda \int_0^\infty h^2 \omega_x \phi \, dx \leq -2\alpha\lambda \int_0^\infty |h h_x \omega| \phi \, dx \leq \frac{1}{2} J(t) + 2\alpha^2 \lambda^2 I(t).$$

Since $\kappa \leq \frac{1}{8}$ and ϕ is supported in $(0, 1)$, it is easy to check that $\omega \leq 2\omega_x$. We get

$$-(\alpha\lambda + \beta + \sqrt{2\kappa}) \int_0^\infty h^2 \omega_x \phi \, dx \leq \frac{1}{2} J(t) + 2\alpha^2 \lambda^2 I(t) - \frac{\beta}{2} I(t).$$

The third term on the right-hand side of (3.5) satisfies

$$-(2\lambda + \kappa) \int_0^\infty h^2 \omega \phi \, dx \leq -2\lambda \int_0^\infty h^2 \omega \phi \, dx \leq \frac{\lambda^2}{2e\kappa} + 2e\kappa I(t)^2.$$

Now we consider the fourth and fifth terms in (3.5). We claim that: there exists a constant C (independent of α, β) such that for any $\varepsilon > 0$ we have

$$(3.6) \quad (\alpha\lambda + \beta) \int_0^\infty h^2 \omega \phi_x \, dx \leq C\varepsilon(I(t) + J(t)) + \frac{C\alpha^2 \lambda^2}{\varepsilon} I(t) + \frac{C\alpha^2 \lambda^2}{\varepsilon},$$

and

$$(3.7) \quad \frac{1}{2} \int_0^\infty h^2 \omega \phi_{xx} dx \leq C\varepsilon(I(t) + J(t)) + \frac{C}{\varepsilon}.$$

Assume the claim holds, and then it follows from (3.5) and the above estimates that there exists $C \geq 1$ independent of α, β such that for any $\varepsilon \in (0, 1)$,

$$I'(t) \leq -\frac{1}{2}J(t) - \kappa I(t) + C\varepsilon(I(t) + J(t)) + \frac{C}{\varepsilon}\alpha^2\lambda^2(I(t) + 1) + \frac{C}{\varepsilon} - \frac{\lambda^2}{2\varepsilon\kappa} + 2\varepsilon\kappa I(t)^2 - \frac{\beta}{2}I(t),$$

which, by taking $\varepsilon := \min\{\frac{1}{2C}, \frac{\kappa}{C}\}$, implies that for $C_1 := \frac{C}{\varepsilon} \geq 1$ (which is independent of α, β , and t), we have

$$(3.8) \quad I'(t) \leq \lambda^2\left(-\frac{1}{2\varepsilon\kappa} + C_1\alpha^2(I(t) + 1)\right) + C_1 - \frac{\beta}{4}I(t) + I(t)\left(2\varepsilon\kappa I(t) - \frac{\beta}{4}\right).$$

Set $M := \max\{1, I(0)\}$ and pick

$$C_0 := \max\{4C_1, 8\varepsilon\kappa M, \sqrt{4\varepsilon\kappa C_1(M + 1)}\} \geq 1,$$

and so $\beta \geq C_0$ and $\alpha \leq 1/C_0$ imply that

$$C_1 - \frac{\beta M}{4} \leq 0, \quad 2\varepsilon\kappa M - \frac{\beta}{4} \leq 0 \quad \text{and} \quad -\frac{1}{2\varepsilon\kappa} + C_1\alpha^2(M + 1) \leq -\frac{1}{4\varepsilon\kappa}.$$

Then it is easy to see from the differential inequality (3.8) that $I(t)$ cannot exceed M . Furthermore, with these choices of α and β , the differential inequality (3.8) yields $I'(t) \leq -\frac{1}{4\varepsilon\kappa}\lambda^2 + C_1$. Hence it follows the global L^2 bound of λ :

$$\int_0^t \lambda^2(s) ds \leq 4\varepsilon\kappa(C_1 t + M)$$

for all $t > 0$ such that the solution is well defined. We conclude by applying Lemma 3.1.

Let us now prove the estimates (3.6)–(3.7). The proof follows closely the arguments in [12]. To show (3.6), since $\beta \geq 0, \lambda \leq 0$, and $\phi_x \leq 0$, it suffices to estimate $\alpha\lambda \int_0^\infty h^2 \omega \phi_x dx$. Consider two nonnegative functions $\varphi_1, \varphi_2 \in C^\infty(0, \infty)$ with $\varphi_1 + \varphi_2 = 1$ such that φ_1 is monotone nonincreasing with $\varphi_1 = 0$ on $(\frac{1}{4}, \infty)$, and φ_2 is monotone nondecreasing with $\varphi_2 = 0$ on $(\frac{1}{8}, \infty)$. With this partition of unity, we can write $\int_0^\infty h^2 \omega \phi_x dx = \int_0^\infty h^2 \omega \phi_x \varphi_1 dx + \int_0^\infty h^2 \omega \phi_x \varphi_2 dx$. Using $0 \leq -\phi_x \leq \phi$ on the support of φ_1 by the first property given in Lemma 3.2, we find

$$-\int_0^\infty h^2 \omega \phi_x \varphi_1 dx \leq \int_0^\infty h^2 \omega \phi dx = I(t).$$

Hence for all $\varepsilon \in (0, 1)$,

$$\alpha\lambda \int_0^\infty h^2 \omega \phi_x \varphi_1 dx \leq \left(\varepsilon + \frac{\alpha^2\lambda^2}{\varepsilon}\right)I(t).$$

For the other part with cut-off function φ_2 , since $\int_0^\infty p(t, x) dx = 1$,

$$\begin{aligned} -\int_0^\infty h^2 \omega \phi_x \varphi_2 dx &= -\int_0^\infty h \partial_x \left(\int_0^x p(t, y) dy\right) \phi_x \varphi_2 dx \\ &\leq \int_0^\infty |h_x| \phi_x \varphi_2 dx + \int_0^\infty h |(\phi_x \varphi_2)_x| dx. \end{aligned}$$

Young's inequality and the fact that ϕ is supported in $(0, 1)$ yield

$$\begin{aligned} -\alpha\lambda \int_0^\infty |h_x|\phi_x\varphi_2 dx &\leq \varepsilon \int_0^\infty h_x^2(\phi_x\varphi_2)^2 dx + \frac{1}{\varepsilon}\alpha^2\lambda^2 \\ &\leq C\varepsilon J(t) + \frac{1}{\varepsilon}\alpha^2\lambda^2. \end{aligned}$$

In the second inequality, we used $(\phi_x\varphi_2)^2 \leq C\phi$ and $\omega(x) \leq \omega(1)$ on $(0, 1)$. Similarly

$$-\alpha\lambda \int_0^\infty h|(\phi_x\varphi_2)_x| dx \leq C\varepsilon I(t) + \frac{1}{\varepsilon}\alpha^2\lambda^2.$$

Putting these estimates together proves (3.6). As for (3.7), due to the third property stated in Lemma 3.2, we have

$$\begin{aligned} \int_0^\infty h^2\omega\phi_{xx} dx &= \int_{x_0}^1 h^2\omega\phi_{xx} dx = \int_{x_0}^1 h\partial_x\left(\int_0^x p(t, y) dy\right)\phi_{xx} dx \\ &= -\int_{x_0}^1 h_x\left(\int_0^x p(t, y) dy\right)\phi_{xx} dx - \int_{x_0}^1 h\left(\int_0^x p(t, y) dy\right)\phi_{xxx} dx \\ &\leq \int_{x_0}^1 |h_x||\phi_{xx}| dx + \int_{x_0}^1 h|\phi_{xxx}| dx, \end{aligned}$$

where in the third equality $\phi_{xx}(x_0) = 0$ is applied, and in the inequality, we used $\int_0^\infty p(t, y) dy = 1$. Due to the second property of Lemma 3.2 and $\omega > 0$ on $[x_0, 1]$, we obtain

$$\int_0^\infty h^2\omega\phi_{xx} dx \leq C \int_0^\infty |h_x|\sqrt{\omega\phi} dx + C \int_0^\infty h\sqrt{\omega\phi} dx \leq C\varepsilon(I(t) + J(t)) + \frac{C}{\varepsilon}$$

which finishes the proof of (3.7). \square

Acknowledgments. The authors would like to thank two anonymous referees, an Associate Editor and the Editor for their constructive comments that improved the quality of this paper.

Funding. E. Bayraktar is partially supported by the National Science Foundation under grant DMS-2106556 and by the Susan M. Smith chair.

W. Tang gratefully acknowledges financial support through an NSF grants DMS-2113779 and DMS-2206038, and through a start-up grant at Columbia University.

Y.P. Zhang acknowledges partial support by an AMS-Simons Travel Grant.

REFERENCES

- [1] ANDREUCCI, D. (2004). Lecture Notes on the Stefan problem. Available at http://www.sbai.uniroma1.it/publicazioni/doc/phd_quaderni/02-1-and.pdf.
- [2] BAKER, G. and SHKOLNIKOV, M. (2022). Zero kinetic undercooling limit in the supercooled Stefan problem. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** 861–871. MR4421610 <https://doi.org/10.1214/21-aihp1194>
- [3] BAYRAKTAR, E., GUO, G., TANG, W. and ZHANG, Y. P. (2023). Systemic robustness: a mean-field particle system approach. Preprint. Available at [arXiv:2212.08518](https://arxiv.org/abs/2212.08518).
- [4] BENEDETTO, D., CAGLIOTI, E., CARRILLO, J. A. and PULVIRENTI, M. (1998). A non-Maxwellian steady distribution for one-dimensional granular media. *J. Stat. Phys.* **91** 979–990. MR1637274 <https://doi.org/10.1023/A:1023032000560>
- [5] BOLLEY, F., GENTIL, I. and GUILLIN, A. (2013). Uniform convergence to equilibrium for granular media. *Arch. Ration. Mech. Anal.* **208** 429–445. MR3035983 <https://doi.org/10.1007/s00205-012-0599-z>
- [6] BURGER, M., CAPASSO, V. and MORALE, D. (2007). On an aggregation model with long and short range interactions. *Nonlinear Anal. Real World Appl.* **8** 939–958. MR2307761 <https://doi.org/10.1016/j.nonrwa.2006.04.002>

- [7] CÁCERES, M. J., CARRILLO, J. A. and PERTHAME, B. (2011). Analysis of nonlinear noisy integrate & fire neuron models: Blow-up and steady states. *J. Math. Neurosci.* **1** Art. 7, 33 pp. MR2853216 <https://doi.org/10.1186/2190-8567-1-7>
- [8] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications. I: Mean Field FBSDEs, Control, and Games. Probability Theory and Stochastic Modelling* **83**. Springer, Cham. MR3752669
- [9] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications. II: Mean Field Games with Common Noise and Master Equations. Probability Theory and Stochastic Modelling* **84**. Springer, Cham. MR3753660
- [10] CARMONA, R., DELARUE, F. and LACHAPELLE, A. (2013). Control of McKean–Vlasov dynamics versus mean field games. *Math. Financ. Econ.* **7** 131–166. MR3045029 <https://doi.org/10.1007/s11579-012-0089-y>
- [11] CARRILLO, J. A., GONZÁLEZ, M. D. M., GUALDANI, M. P. and SCHONBEK, M. E. (2013). Classical solutions for a nonlinear Fokker–Planck equation arising in computational neuroscience. *Comm. Partial Differential Equations* **38** 385–409. MR3019444 <https://doi.org/10.1080/03605302.2012.747536>
- [12] CARRILLO, J. A., PERTHAME, B., SALORT, D. and SMETS, D. (2015). Qualitative properties of solutions for the noisy integrate and fire model in computational neuroscience. *Nonlinearity* **28** 3365–3388. MR3403402 <https://doi.org/10.1088/0951-7715/28/9/3365>
- [13] CARSLAW, H. S. (1945). *Introduction to the Mathematical Theory of the Conduction of Heat in Solids*. Dover, New York. MR0015635
- [14] CUCHIERO, C., RIGGER, S. and SVALUTO-FERRO, S. (2023). Propagation of minimality in the supercooled Stefan problem. *Ann. Appl. Probab.* **33** 1388–1418. MR4564435 <https://doi.org/10.1214/22-aap1850>
- [15] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Global solvability of a networked integrate-and-fire model of McKean–Vlasov type. *Ann. Appl. Probab.* **25** 2096–2133. MR3349003 <https://doi.org/10.1214/14-AAP1044>
- [16] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Particle systems with a singular mean-field self-excitation. Application to neuronal networks. *Stochastic Process. Appl.* **125** 2451–2492. MR3322871 <https://doi.org/10.1016/j.spa.2015.01.007>
- [17] DELARUE, F., NADTOCHIY, S. and SHKOLNIKOV, M. (2022). Global solutions to the supercooled Stefan problem with blow-ups: Regularity and uniqueness. *Probab. Math. Phys.* **3** 171–213. MR4420299 <https://doi.org/10.2140/pmp.2022.3.171>
- [18] FASANO, A. and PRIMICERIO, M. (1980/81). New results on some classical parabolic free-boundary problems. *Quart. Appl. Math.* **38** 439–460. MR0614552 <https://doi.org/10.1090/qam/614552>
- [19] FASANO, A. and PRIMICERIO, M. (1983). A critical case for the solvability of Stefan-like problems. *Math. Methods Appl. Sci.* **5** 84–96. MR0690897 <https://doi.org/10.1002/mma.1670050107>
- [20] FRIEDMAN, A. (1959). Free boundary problems for parabolic equations. I. Melting of solids. *J. Math. Mech.* **8** 499–517. MR0144078 <https://doi.org/10.1512/iumj.1959.8.58036>
- [21] HAMBLY, B., LEDGER, S. and SØJMARK, A. (2019). A McKean–Vlasov equation with positive feedback and blow-ups. *Ann. Appl. Probab.* **29** 2338–2373. MR3983340 <https://doi.org/10.1214/18-AAP1455>
- [22] HUANG, M., MALHAMÉ, R. P. and CAINES, P. E. (2006). Large population stochastic dynamic games: Closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.* **6** 221–251. MR2346927
- [23] KAC, M. (1956). Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, Vol. III* 171–197. Univ. California Press, Berkeley–Los Angeles, CA. MR0084985
- [24] KELLER, E. F. and SEGEL, L. A. (1971). Model for chemotaxis. *J. Theoret. Biol.* **30** 225–234.
- [25] KNERR, B. F. (1980/81). Parabolic interior Schauder estimates by the maximum principle. *Arch. Ration. Mech. Anal.* **75** 51–58. MR0592103 <https://doi.org/10.1007/BF00284620>
- [26] LADYŽENSKAJA, O. A., SOLONNIKOV, V. A. and URAL’CEVA, N. N. (1968). *Linear and Quasilinear Equations of Parabolic Type. Translations of Mathematical Monographs* **23**. Amer. Math. Soc., Providence, RI. MR0241822
- [27] LEDGER, S. and SØJMARK, A. (2020). Uniqueness for contagious McKean–Vlasov systems in the weak feedback regime. *Bull. Lond. Math. Soc.* **52** 448–463. MR4171379 <https://doi.org/10.1112/blms.12337>
- [28] LEDGER, S. and SØJMARK, A. (2021). At the mercy of the common noise: Blow-ups in a conditional McKean–Vlasov problem. *Electron. J. Probab.* **26** Paper No. 35, 39 pp. MR4235486 <https://doi.org/10.1214/21-EJP597>
- [29] LIEBERMAN, G. M. (1996). *Second Order Parabolic Differential Equations*. World Scientific Co., Inc., River Edge, NJ. MR1465184 <https://doi.org/10.1142/3302>

- [30] MCKEAN, H. P. JR. (1966). A class of Markov processes associated with nonlinear parabolic equations. *Proc. Natl. Acad. Sci. USA* **56** 1907–1911. [MR0221595](#) <https://doi.org/10.1073/pnas.56.6.1907>
- [31] MEI, S., MONTANARI, A. and NGUYEN, P.-M. (2018). A mean field view of the landscape of two-layer neural networks. *Proc. Natl. Acad. Sci. USA* **115** E7665–E7671. [MR3845070](#) <https://doi.org/10.1073/pnas.1806579115>
- [32] NADTOCHIY, S. and SHKOLNIKOV, M. (2019). Particle systems with singular interaction through hitting times: Application in systemic risk modeling. *Ann. Appl. Probab.* **29** 89–129. [MR3910001](#) <https://doi.org/10.1214/18-AAP1403>
- [33] PACHPATTE, B. G. (1998). *Inequalities for Differential and Integral Equations. Mathematics in Science and Engineering* **197**. Academic Press, San Diego, CA. [MR1487077](#)
- [34] ROTSKOFF, G. M. and VANDEN-EIJNDEN, E. (2022). Trainability and accuracy of artificial neural networks: An interacting particle system approach. *Comm. Pure Appl. Math.* **75** 1889–1935. [MR4465905](#)
- [35] STRAUSS, W. A. (2008). *Partial Differential Equations: An Introduction*, 2nd ed. Wiley, Chichester. [MR2398759](#)