COMPUTING LAPLACE TRANSFORMS FOR NUMERICAL INVERSION VIA CONTINUED FRACTIONS

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***Abstract*—It is often possible to effectively calculate probability density functions (pdf's) and cumulative distribution functions (cdf's) by numerically inverting Laplace transforms. However, to do so it is necessary to compute the Laplace transform values. Unfortunately, convenient explicit expressions for required transforms are often unavailable for component pdf's in a probability model. In that event, we show that it is sometimes possible to find continued-fraction representations for required Laplace transforms that can serve as a basis for computing the transform values needed in the inversion algorithm. This property is very likely to prevail for completely monotone pdf's, because their Laplace transforms have special continued fractions called S fractions, which have desirable convergence properties. We illustrate the approach by considering applications to compute first-passage-time cdf's in birth-and-death processes and various cdf's with non-exponential tails, which can be used to model service-time cdf's in queueing models. Included among these cdf's is the Pareto cdf.**

***Keywords*—computational probability, numerical transform inversion, continued fractions, Laplace transforms, S fractions, complete monotonicity, Padé approximants, cumulative distribution function, birth-and-death process, Pareto distribution.**

Many probability density functions (pdf's) and cumulative distribution functions (cdf's) of interest in queueing models and other probability models arising in operations research can be effectively computed by numerically inverting Laplace transforms; see Abate, Choudhury and Whitt [1], Abate and Whitt [4], [5] and references therein. The biggest challenge in this approach, when there is a challenge, is usually computing the required Laplace transform values, because convenient closed-form expressions for Laplace transforms often are not available. In this paper we point out that continued fractions can sometimes serve as a basis for effectively computing the required Laplace transform values needed in the inversion algorithms.

A simple motivating example is the steady-state waiting-time pdf in the M/G/1 queue. The classical Pollaczek-Khintchine (transform) formula gives the Laplace transform of the steady-state waiting-time pdf in terms of the Laplace transform of the service-time pdf. Thus we can compute the waiting-time transform values in order to compute the waiting-time pdf or cdf by numerical inversion whenever we can compute the service-time transform values. A possible difficulty, however, is that we might want to consider service-time pdf's for which convenient explicit expressions for the Laplace transform are unavailable. Indeed, this difficulty often arises when we consider distributions which have non-exponential tails, e.g., which cannot be represented as phase-type distributions. The present paper provides a way to address this problem: Under favorable circumstances, we may be able to construct a continued-fraction representation of the service-time Laplace transform that enables us to compute the service-time Laplace transform values, which in turn enables us to compute the waiting-time Laplace transform values needed to perform the desired numerical inversion. A specific example covered by this approach is the Pareto pdf.

For background on continued fractions and their use for numerical computation, see Baker and Graves-Morris [12], Bender and Orszag [13], Chapter 12 of Henrici [26], Jones and Thron [28], Section 5.2 of Press, Flannery, Teukolsky and Vetterling [32] and Wall [35]. Applications of continued fractions in statistics and applied probability are described in Bowman and Shenton [15] and Bordes and Roehner [14]. More recently, Guillemin and Pinchon [20], [21], [22], [23] have used continued fractions to analytically derive important properties of queueing models. A summary of that work is contained in Dupuis and Guillemin [16]. However, continued fractions evidently have not been suggested previously as a way to numerically compute transform values in order to perform numerical transform inversion.

The use of continued fractions is an alternative to computation of Laplace transforms via infinite-series representations, which we recently discussed in Abate and Whitt [9]. We make an explicit numerical comparison to show that continued fractions can be far superior in some circumstances, even when the series converges geometrically. (See Section 6.)

Here is how the rest of this paper is organized: In Section 1 we briefly define continued fractions and specify the basic recursive algorithm for numerical computation. In Section 2 we discuss the relation between continued fractions and power series. There we show how to compute the continued fraction elements from the moments of a probability distribution (which are related to the coefficients of a power series — the moment generating function). In Section 3 we point out that completely monotone pdf's can be identified with special continued fractions called S fractions, which have nice convergence properties. In Section 4 we show how continued fractions can be used to compute the Laplace transforms of first-passage-times pdf's in birth-and-death processes. We can exploit S fractions for this purpose because first passage times to neighboring states have completely monotone pdf's.

Subject classifications: Probability distributions: calculation by transform inversion. Queues, algorithms: Laplace transform inversion. mathematics, functions: Laplace transforms and continued fractions

The rest of the paper is devoted to numerical examples. In Section 5 we consider the M/M/∞ busy period, which is a special case of a first passage time in a birth-and-death process. In Section 6 we consider the beta mixture of exponential (BME) pdf's from Abate and Whitt [8] [9] and show that continued fractions can be much more effective for computing Laplace transform values than the previously considered infinite-series representations. In Section 7 we show how the continued fractions associated with the BME pdf's can be used to compute the Laplace transforms of other pdf's related to the BME pdf's, including a Pareto pdf. Finally, in Section 8 we discuss continued fraction representations of Laplace transforms of other pdf's.

1. Continued Fractions

An (infinite) *continued fraction* (CF) associated with a sequence $\{a_n : n \geq 1\}$ of *partial numerators* and a sequence $\{b_n : n \geq 1\}$ of *partial denominators*, which are complex numbers with $a_n \neq 0$ for all n, often called *elements*, is the sequence $\{w_n : n \geq 1\}$, where

$$w_n = t_1 \circ t_2 \circ \ldots \circ t_n(0), \quad n \geq 1 \ , \tag{1.1}$$

and

$$t_k(u) = \frac{a_k}{b_k + u}, \quad k \geq 1 \ ; \tag{1.2}$$

i.e., w_n is the n-fold composition of the mappings $t_k(u)$ in (1.2) applied to 0, called the n^{th} *approximant*. If $\lim_{n\to\infty} w_n = w$, then the CF is said to be (properly) *convergent* and the limit w is called the *value* of the CF. We write

$$w = \Phi_{n=1}^{\infty} \frac{a_n}{b_n} \quad \text{or} \quad w = \frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\ \frac{a_3}{b_3+} \cdots \ . \tag{1.3}$$

When we consider Laplace transforms of pdf's, the CF elements a_n and b_n will be functions of the complex variable s. In particular, we will consider special CFs called S fractions (S for Stieltjes), which can be expressed as

$$w \equiv w(s) = \frac{1}{1+}\ \frac{a_2 s}{1+}\ \frac{a_3 s}{1+}\ \frac{a_4 s}{1+} \cdots \tag{1.4}$$

where a_k is real and positive for all k. However, S fractions may appear in other forms, because CFs have many equivalent representations. Indeed, for any sequence of complex numbers $\{c_n : n \geq 0\}$ with $c_0 = 1$, the CF

$$\Phi_{n=1}^{\infty} \frac{c_{n-1} c_n a_n}{c_n b_n} \tag{1.5}$$

has the same approximants as the CF in (1.3); see p. 478 of Henrici [26]. We call such CFs *equivalent* and use the notation $\simeq$.

It is significant that there is a relatively simple recursion for calculating the successive approximants of a CF, due to Euler in 1737. In particular, given the CF in (1.3),

$$w_n = \frac{P_n}{Q_n} \ , \tag{1.6}$$

where $P_0 = 0$, $P_1 = a_1$, $Q_0 = 1$, $Q_1 = b_1$ and

$$\begin{aligned} P_n &= b_n P_{n-1} + a_n P_{n-2} \\ Q_n &= b_n Q_{n-1} + a_n Q_{n-2} \end{aligned} \tag{1.7}$$

for $n \geq 2$. In performing numerical calculations, it is prudent to renormalize after, say, every 10 iterations by dividing the current values of P_k, Q_k, P_{k-1} and Q_{k-1} all by Q_k, and then proceed.

We will be interested in the special case of S fractions, as in (1.4). Based on our computational experience, we conclude that the S fraction converges rapidly and is easy to compute if $a_n = O(1)$ as $n \to \infty$. If $a_n = O(n)$ as $n \to \infty$, then the S fraction converges, but more slowly and requires more work to calculate. If the elements grow much faster, then computation is likely to be infeasible. (We will give examples later.) The S fraction with $a_n = n^k$ can be shown to be convergent if and only if $k \leq 2$; see p. 486 of Henrici [26].

2. Power Series and Continued Fractions

Continued fractions are intimately related to power series. This relationship is useful in probability applications because the moments of probability distributions can be regarded as coefficients of a power series, namely, the *moment generating function* (mgf).

Let f be a pdf on the nonnegative real line with associated cdf $F(t) \equiv \int_0^t f(u)du$, $t \geq 0$, and associated *complementary cdf* (ccdf) $F^c(t) \equiv 1 - F(t)$, $t \geq 0$. Assume that the pdf f has finite moments of all orders, i.e.,

$$m_n(f) \equiv \int_0^{\infty} t^n f(t)dt, \quad n \geq 1 \ , \tag{2.1}$$

and let $\hat{f}$ be the *Laplace transform* (LT) of f, i.e.,

$$\hat{f}(s) \equiv \int_0^{\infty} e^{-st} f(t)dt \ , \tag{2.2}$$

where s is a complex variable. We often are most interested in computing the cdf $F(t)$ or ccdf $F^c(t)$, which we can do via their LTs $\hat{F}(s) = \hat{f}(s)/s$ and $\hat{F^c}(s) = (1 - \hat{f}(s))/s$. The associated mgf is $\hat{f}(-s)$.

The Laplace transform $\hat{f}$ is analytic for all s with $\text{Re}(s) > 0$. The n^{th} moment of f can be recovered from the n^{th} derivative of the transform, i.e.,

$$m_n(f) = -\frac{d^{(n)}}{ds} \hat{f}(s)\Big|_{s=0} \ , \tag{2.3}$$

where limits are taken through real positive s, and the LT can be represented as a *formal power* (Maclaurin or Taylor) *series*

$$\hat{f}(s) = \sum_{n=0}^{\infty} c_n s^n \ , \tag{2.4}$$

where $c_0 = 1$ and

$$c_n = (-1)^n \frac{m_n(f)}{n!} \ , \quad n \geq 1 \ , \tag{2.5}$$

by which we mean that, for each $N \geq 1$,

$$\hat{f}(s) - \sum_{n=0}^{N} c_n s^n = O(s^{N+1}) \quad \text{as} \quad s \to 0 \ . \tag{2.6}$$

However, in general we cannot conclude that the power series (2.4) has a positive radius of convergence. It will if and only if the LT is analytic at 0, which is not always the case.

Given a formal power series such as (2.4), we can construct rational approximants, called *Padé approximants*, that match the coefficients of (2.4) as far as possible; see Chapter 1 of Baker and Graves-Morris [12]. The $[L/M]$ Padé approximant to $\hat{f}$ is the rational function

$$[L/M] \equiv \frac{a_0 + a_1 s + \cdots + a_L s^L}{1 + b_1 s + \cdots + b_M s^M} \ , \tag{2.7}$$

where the first $L+M+1$ coefficients of the Maclaurin series of (2.7) match the first $L+M+1$ coefficients in (2.4).

Given a formal power series such as (2.4), it is also possible to construct an associated CF of the form

$$\hat{f}(s) = a_0 + \Phi_{n=1}^{\infty} \frac{a_n s}{1} \ , \tag{2.8}$$

whose approximants also have Maclaurin series that match the initial terms of the power series (2.4). (A CF with the variable s in each coefficient as in (2.8) is sometimes called a RITZ fraction; e.g., see p. 515 of Henrici [26].) To obtain the CF representation, we use the notion of the reciprocal of a power series to create an appropriate recursion. In particular, we consider the reciprocal of the series

$$1 + \frac{c_2 s}{c_1} + \frac{c_3 s^2}{c_1} + \cdots = (1 + c_1^{(1)} s + c_2^{(1)} s^2 + \cdots)^{-1} \tag{2.9}$$

to obtain

$$\sum_{n=0}^{\infty} c_n s^n = c_0 + \frac{c_1}{1 + c_1^{(1)} s + c_2^{(1)} s^2 + \cdots} \tag{2.10}$$

from (2.4). Next consider the reciprocal of the series

$$1 + \frac{c_2^{(1)} s}{c_1^{(1)}} + \frac{c_3^{(1)} s^2}{c_1^{(1)}} + \cdots = (1 + c_1^{(2)} s + c_2^{(2)} s^2 + \cdots)^{-1} \tag{2.11}$$

to obtain

$$\sum_{n=0}^{\infty} c_n s^n = c_0 + \frac{c_1 s}{1 + \frac{c_1^{(1)} s}{1 + c_1^{(2)} s + c_2^{(2)} s^2 + \cdots}} \ . \tag{2.12}$$

Proceeding by induction, we obtain (2.8) with $a_0 = c_0$, $a_1 = c_1$ and $a_n = c_1^{(n-1)}$, $n \geq 2$.

Since the approximants of the CF in (2.8) are rational functions, it should come as no surprise that there is a link between the CF in (2.8) and Padé approximants. It turns out that the CF approximants w_{2M} and w_{2M+1} are "diagonal" Padé approximants; in particular, they are precisely the $[M/M]$ and $[M+1/M]$ Padé approximants, respectively; see Theorem 4.2.1 of Baker and Graves-Morris [12]. (If the CF is terminating, corresponding to a rational function, then the approximants may be of lower order.)

Thus, methods for computing diagonal Padé approximants are equivalent to methods for computing approximants for CFs associated with power series. A powerful computational procedure is the *quotient-difference* (QD) *algorithm*; see Chapter 3 and Section 4.3 of Baker and Graves-Morris [12]. The QD algorithm has $a_1 = c_0$, $a_{2k} = -q_k^0$ and $a_{2k+1} = -e_k^0$, $k \geq 1$, where q_k^j and e_k^j are defined recursively by

$$q_{k+1}^j = e_k^{j+1} q_k^{j+1} / e_k^j \tag{2.13}$$

and

$$e_k^j = e_{k-1}^{j+1} + q_k^{j+1} - q_k^j \tag{2.14}$$

for $k \geq 1$ and $j \geq 0$, with $e_0^j = 0$, $j \geq 1$, and $q_1^j = c_{j+1}/c_j$, $j \geq 0$. The QD algorithm is related to the ϵ-algorithm; e.g., see Chapter 3 of Baker and Graves-Morris [12] and Chapter 8 of Wimp [37].

An essentially equivalent computational procedure is the *product-difference* (PD) *algorithm* due to Gordon [19]. The PD algorithm involves only a single array p_i^j, instead of the two arrays e_i^j and q_i^j in the QD algorithm. In particular, the recursion is

$$p_i^j = p_1^{j-1} p_{i+1}^{j-2} - p_1^{j-2} p_{i+1}^{j-1} \ , \tag{2.15}$$

where, for $i \geq 1$, p_i^1 and p_i^2 are initialized to $p_1^1 = 1$, $p_i^1 = 0$ for $i \geq 2$, and $p_i^2 = c_{i-1}$ for $i \geq 1$. Then

$$a_n = \frac{p_1^{n+1}}{p_1^{n-1} p_1^n} \ . \tag{2.16}$$

Hence, given the moments of a pdf as in (2.1), we can obtain a CF representation of its LT in (2.2), with the property that the CF approximants coincide with diagonal Padé approximants. We now discuss additional structure that ensures that the CF is actually convergent.

3. Complete Monotonicity and S Fractions

Probability applications of CFs are especially appealing when we have a completely monotone (CM) pdf because then the associated LTs can be represented by special CFs called S fractions, which are known to converge under minor regularity conditions. A CF of the form (2.8) is an *S fraction* if all of the coefficients a_n (not considering the complex variable s) are positive. With such a simple characterization, it is also often possible to verify that a CF is an S fraction directly.

A function f on $[0, \infty)$ is *completely monotone* (CM) if it possesses derivatives of all orders that alternate in sign, i.e.,

$$(-1)^n f^{(n)}(t) \geq 0 \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad n \geq 0 \ ; \tag{3.1}$$

see p. 439 of Feller [18] and p. 66 of Keilson [29]. All CM pdf's are *log convex*, i.e., $\log f(t)$ is convex, and thus have decreasing failure rate (are DFR); see p. 74 of Keilson [29]. The family of CM pdf's is closed under mixtures. A pdf f on $[0,\infty)$ is CM if and only if f is a mixture of exponential pdf's, i.e.,

$$f(t)=\int_0^\infty y^{-1}e^{-t/y}dH(y) \qquad (3.2)$$

for some cdf H on $[0,\infty)$. Examples of CM pdf's appear in Abate and Whitt [2], [6], [8] and references therein.

By making the change of variables $x=y^{-1}$, we see that the CM pdf f in (3.2) can also be represented as

$$f(t)=\int_0^\infty xe^{-xt}d\tilde{H}(x)\ , \qquad (3.3)$$

where $d\tilde{H}(x)=x^{-2}dH(x^{-1})$ and $dH(x)=x^{-2}d\tilde{H}(x^{-1})$. We call H in (3.2) the *mixing cdf* and $\tilde{H}$ in (3.3) the *spectral cdf*. (The mixing and spectral cdf's for the M/M/1 busy period are displayed in [2].) Let X be a mean-1 exponential random variable; let Y and $\tilde{Y}$ be random variables with cdf's H and $\tilde{H}$, respectively. Then the representations (3.2) and (3.3) are the pdf's of XY and $X/\tilde{Y}$, respectively.

The CM pdf f in (3.2) has Laplace transform

$$\hat{f}(s)\equiv\int_0^\infty e^{-st}f(t)dt=\int_0^\infty(1+sy)^{-1}dH(y)\ . \qquad (3.4)$$

Associated with any cdf F is a ccdf $F^c(t)\equiv 1-F(t)$. Associated with the CM pdf f is a *dual ccdf*

$$G^c(t)=\int_0^\infty e^{-xt}dH(x), t\geq 0\ , \qquad (3.5)$$

and associated dual pdf

$$g(t)=\int_0^\infty xe^{-xt}dH(x), t\geq 0\ , \qquad (3.6)$$

which are obtained by switching the roles of H and $\tilde{H}$. Clearly, $G^c(t)$ and $g(t)$ are CM too. It is evident from (3.4) and (3.5) that the dual Laplace transforms are related by

$$\begin{aligned}\hat{G}^c(s) &\equiv \int_0^\infty e^{-st}G^c(t)dt\\ &= \int_0^\infty(s+x)^{-1}dH(x)=s^{-1}\hat{f}(s^{-1})\ . \quad (3.7)\end{aligned}$$

Note that the moments of the CM pdf f in (3.2) are related to the moments of the mixing cdf H by

$$m_n(f)=m_n(H)n! \quad \text{for all} \quad n\geq 1\ . \qquad (3.8)$$

Thus if f has moments of all orders, then its LT has the (formal) power series representation

$$\begin{aligned}\hat{f}(s) &= \sum_{n=0}^\infty\frac{\hat{f}^n(0)s^n}{n!}\\ &= \sum_{n=0}^\infty\frac{m_n(F)(-s)^n}{n!}=\sum_{n=0}^\infty m_n(H)(-s)^n\ , \quad (3.9)\end{aligned}$$

where $\hat{f}^{(0)}(0)\equiv\hat{f}(0)\equiv m_0(F)\equiv m_0(H)=1$. If the LT $\hat{f}(s)$ is analytic at 0, then the power series (3.9) has a positive radius of convergence and is not formal, but it suffices to have only a formal power series.

We now relate these probabilistic quantities to continued fractions. The main connection is that, under the complete monotonicity assumption, the CF associated with the power series of $\hat{f}(s)$ in (3.9) is an S fraction; see Chapter 5 of Baker and Graves-Morris [12]. Except for the normalization $\hat{f}(0)=1$, which holds for pdf's, Laplace transforms of CM pdf's with all moments finite, having the integral representation (3.4), coincide with *Stieltjes functions* in the theory of continued fractions; see p. 193 of Baker and Graves-Morris [12]. As a regularity condition to avoid the case of terminating CFs, it is usually assumed that the underlying cdf H has infinitely many points of increase (does not have finite support). The associated series $\sum_{n=0}^\infty m_n(H)(-s)^n$ in (3.9) then is called the associated *Stieltjes series*. The series is called formal because it may not converge for any s (except $s=0$). In Section 2 we saw that there is a CF with denominator elements 1 associated with any formal power series. The fact that we have a Stieltjes series implies that the numerator CF elements (not counting the complex variable s) are all positive, i.e., that we have an S fraction. The advantage of S fractions is that there is more supporting convergence theory.

We now note duality properties of the CM pdf $f(t)$ in (3.2) and the dual ccdf $G^c(t)$ in (3.5). Since the LTs are related by $\hat{G}^c(s)=s^{-1}\hat{f}(s^{-1})$, it is immediate that the CM for $\hat{f}(s)$ converges at s if and only if the CM for $\hat{G}^c(s)$ converges at s^{-1}. We can also relate the two CFs.

Proposition 3.1. *The LT of a CM pdf f has an S fraction representation*

$$\hat{f}(s)=\frac{a_1}{1+}\ \frac{a_2s}{1+}\ \frac{a_3s}{1+}\ \frac{a_4s}{1+}\cdots \qquad (3.10)$$

with $a_1=1$ and $a_k>0$ for all k if and only if the LT of the dual ccdf G^c has the S fraction representation

$$\begin{aligned}\hat{G}^c(s) &= s^{-1}\left(\frac{a_1}{1+}\ \frac{a_2s^{-1}}{1+}\ \frac{a_3s^{-1}}{1+}\cdots\right)\\ &\simeq \frac{a_1}{s+}\ \frac{a_2}{1+}\ \frac{a_3}{s+}\ \frac{a_4}{1+}\cdots \qquad (3.11)\end{aligned}$$

Proof. Since $\hat{G}^c(s)=s^{-1}\hat{f}(s^{-1})$, from (3.10) we immediately see that $\hat{G}^c(s)$ has the first representation in (3.11). Then we obtain the second relation in (3.11) by applying the equivalence transformation in (1.5). ■

The convergence of S fractions is intimately linked to the classical moment problem; see Akhiezer [11], Chapter 5 of Baker and Graves-Morris [12] and Sections 12.7–12.14 of Henrici [26]. For the following result, we apply Theorems 12.14b and 12.8e in Henrici [26]. We also use the fact that the CF associated with $\hat{G}^c(s)$ converges to s if and only if the CF associated with $\hat{f}(s)$ converges at s^{-1}.

Theorem 3.1. *Let f be a CM pdf as in (3.2) with all moments finite. Then the CFs associated with the LTs $\hat{f}$ in*

(3.4) and $\hat{G}^c$ in (3.7) are convergent (uniformly in compact subsets of the complex plane minus the negative real axis) if and only if the moments $m_n(H)$ uniquely determine the cdf H.

Note that an assumption in Theorem 3.1 is that the pdf f in (3.2) has all moments finite. By (3.8), H too necessarily has all moments finite. Note that Theorem 3.1 does not preclude $\hat{f}$ having a convergent S fraction when the pdf f does *not* have all moments finite. Indeed the dual ccdf G^c may fail to have all moments finite when the conditions of Theorem 3.1 are satisfied, so that $\hat{G}^c$ has a convergent S fraction.

It is also possible that the cdf H has all moments finite and is uniquely determined by those moments, but that the cdf F is not uniquely determined by its moments. An example of such an F is the Weibull cdf with exponent 1/2. Thus this Weibull cdf is not uniquely determined by its moments, but its LT nevertheless has a convergent S fraction. (We discuss this example in Section 8.)

When the complex variable s in an S fraction is real and positive, we also have a convenient a posteriori bound on the numerical errors in the approximants, because then the even approximants w_{2m} are increasing, while the odd approximants are decreasing, so that

$$w_{2n}(s) < \hat{f}(s) < w_{2n+1}(s) \quad \text{for all} \quad n ; \qquad (3.12)$$

see Theorem 12.11d of Henrici [26]. Moreover, we can numerically verify convergence for all complex s (except negative real s) by considering the case of any single positive real s, by Theorem 12.8e of [26]. Various a priori and a posteriori bounds on the error are also given in [26].

The fact that property (3.12) holds for all positive real s motivates using inversion methods based on positive real s. One such method is the Gaver-Stehfest inversion algorithm in [4]. However, in our numerical examples we use the Fourier-series method with Euler summation, which is the main method described in [1], [4], [5]. It requires computing Laplace transform values at complex s.

We close this section by showing what happens for a typical diverging S fraction. We let $a_n = (n-1)^3$ for $n \geq 2$. Table 1 shows the values of the even and odd approximants as a function of n for $s = 1$ and $s = 10$. First, monotonicity as in (3.12) and non-convergence are evident from the numerical results. Second, the odd and even approximants converge remarkably slowly to their limits, so that we cannot easily make use of the limits of the odd and even approximants either.

When the power series (3.9) is only formal, it is typically divergent. However, we have seen that the associated CF may nevertheless be convergent. On the other hand, as illustrated by Table 1, the CF may be divergent. When the CF converges, we have a way to sum a divergent series — called Stieltjes summation; see Chapter 19 of Wall [35].

index	$s=1$		$s=10$	
n	even	odd	even	odd
$4,000$	0.6637	0.7927	0.1777	0.7312
$8,000$	0.6643	0.7921	0.1783	0.7301
$12,000$	0.6646	0.7918	0.1785	0.7296
$16,000$	0.6648	0.7917	0.1787	0.7293
$20,000$	0.6649	0.7916	0.1788	0.7291

Table 1: Values of even and odd S fraction approximants for $s = 1$ and 10 when $a_n = (n-1)^3$ for $n \geq 2$.

4. First Passage Times in Birth-and-Death Processes

In this section we show how CFs can be used to compute the LT of a first-passage-time pdf in a birth-and-death (BD) process. Let $T_{i,j}$ be a random variable representing the first passage time from state i to state j. It is elementary that such first passage times can be expressed in terms of first passage times to neighboring states; e.g., if $i < j$, then

$$T_{i,j} = T_{i,i+1} + T_{i+1,i+2} + \cdots + T_{j-1,j} \ , \qquad (4.1)$$

where the random variables on the right are mutually independent, and similarly if $i > j$. Let $f_{i,j}$ be the pdf of $T_{i,j}$ and let $\hat{f}_{i,j}$ be its LT, i.e.,

$$\hat{f}_{i,j}(s) \equiv \int_0^\infty e^{-st} f_{i,j}(t) dt \equiv E e^{-sT_{i,j}} \ . \qquad (4.2)$$

From (4.1), we have

$$\hat{f}_{i,j}(s) = \prod_{k=i}^{k=j-1} \hat{f}_{k,k+1}(s) \qquad (4.3)$$

if $i < j$. Hence, in order to compute the LT $\hat{f}_{i,j}$, it suffices to be able to compute the LT of the first passage time to a neighboring state.

First passage times to neighboring states are especially tractable because their pdf's are always CM. For finite-state BD processes, this CM property is an elementary consequence of the spectral theory associated with these reversible Markov processes; see Section 3.4B of Keilson [29]. For first passage times up in infinite-state BD processes, the states above the destination state play no role, so that the state space may be considered finite. For first passage times down in infinite-state BD processes, the CM property can be deduced by considering the limit of the finite-state approximations for which in the BD process n the original birth rate (in the infinite-state model) in state n, λ_n, is set equal to 0. (The holding time in state n is then exponential with mean μ_n^{-1}, where μ_n is the death rate.)

Let $T_i^{(n)}$ denote the first passage time down from state i to state $i-1$ in BD process n with state space $\{0,1,\ldots,n\}$, where $n > i$. Let $F_i^{(n)}$ be the cdf of $T_i^{(n)}$. By the construction above, we can make sample-path comparisons as in Whitt [36] to deduce that a stochastic order relation holds as we change n, i.e.,

$$T_i^{(n)} \leq_{st} T_i^{(n+1)} \quad \text{for} \quad n \geq i \; , \tag{4.4}$$

by which we mean that

$$F_i^{(n)}(t) \geq F_i^{(n+1)}(t) \quad \text{for all} \quad t \; . \tag{4.5}$$

Hence, $F_i^{(n)}(t)$ decreases to a limit $F_i(t)$ as $n \to \infty$, which need be a proper cdf. We then apply the following result to deduce that the limiting cdf F_i is CM.

Proposition 4.1. *If $\{F_n : n \geq 1\}$ is a sequence of CM cdf's on $[0,\infty)$ such that $F_n(t) \to G(t)$ as $n \to \infty$ for each t, then the limit G is CM. If $F_n^c(0) \to 1$, then G is proper.*

Proof. Recall that the ccdf's F_n^c can be expressed as

$$F_n^c(t) = \int_0^\infty e^{-xt} dH_n(x) \tag{4.6}$$

for cdf's H_n by the CM property, so that the ccdf $F_n^c(t)$ can be identified with the Laplace-Stieltjes transform of the cdf H_n evaluated at $s = t$ for t real and positive. Hence we can apply the continuity theorem for Laplace transforms on p. 431 of Feller [18]. ■

It is also easy to directly construct CFs representing the LTs of first passage times down with an infinite state space. Let λ_i and μ_i denote the birth and death rates in state i, respectively. Let $\hat{f}_i$ denote the LT of the pdf of the first passage time from state i to state $i-1$. By considering the first transition, we obtain the recursion

$$\begin{aligned} \hat{f}_i(s) &= \left(\frac{\mu_i}{\lambda_i+\mu_i}\right)\left(\frac{\lambda_i+\mu_i}{\lambda_i+\mu_i+s}\right) \\ &\quad + \left(\frac{\lambda_i}{\lambda_i+\mu_i}\right)\left[\left(\frac{\lambda_i+\mu_i}{\lambda_i+\mu_i+s}\right)\hat{f}_{i+1}(s)\hat{f}_i(s)\right] \\ &= \frac{\mu_i}{\lambda_i+\mu_i+s} + \frac{\lambda_i \hat{f}_{i+1}(s)\hat{f}_i(s)}{\lambda_i+\mu_i+s} \end{aligned} \tag{4.7}$$

from which we obtain

$$\hat{f}_i(s) = \frac{\mu_i}{\lambda_i+\mu_i+s-\lambda_i\hat{f}_{i+1}(s)} \; . \tag{4.8}$$

Iterating on (4.8) produces the CF

$$\hat{f}_i(s) = -\frac{1}{\lambda_{i-1}} \Phi_{k=i}^{\infty} \frac{-\lambda_{k-1}\mu_k}{\lambda_k+\mu_k+s} \; , \tag{4.9}$$

which directly has the form of a real J fraction (J for Jacobi), which can be shown to be equivalent to an S fraction; see Baker and Graves-Morris [12], Wall [35] and Dupuis and Guillemin [16]. However, the CM property implies the equivalence, so that we do not need to construct one.

The analysis above shows that first passage times to neighboring states have CM pdf's, so that by Section 3 their LTs have CF representations that are S fractions. This S fraction representation provides a basis for computing the LT, which in turn can be used to calculate the more general first-passage-time pdf's and cdf's by numerical transform inversion, using (4.3).

Given the CM transform representation (3.4), if the mixing cdf H has all moments finite, then the CF (4.9) is convergent if and only if the moments of H uniquely determine H by Theorem 3.1. As noted at the end of Section 3, numerically, convergence for all s (except negative real s) can be verified by considering the case of a single real s. Then there is convergence if and only if the gap between odd and even approximants i.e., $w_{2n+1}(s) - w_{2n}(s)$ in (3.12) decreases to 0 as n increases.

In queueing applications the first passage time of greatest interest is T_{10}, which corresponds to the busy period. Interesting special cases are the M/M/1, M/M/s, M/M/∞, M/M/s/0 and M/M/s/r systems. It is interesting that the mixing and spectral cdf's have a continuous spectrum (interval of support) in the M/M/1 case, see Abate and Whitt [2], but have a countably infinite spectrum (support) in the M/M/∞ case; see the next section. Alternative methods for computing the first-passage-time LTs in the M/M/s/0 case were recently discussed in Abate and Whitt [7].

5. The M/M/∞ Busy Period

In this section we apply the CF for BD first-passage-time LTs in Section 4 to calculate the busy-period ccdf in the M/M/∞ queue. We apply the Fourier-series method with Euler summation, the algorithm EULER in [5], to numerically invert the LT after we calculate the required LT values.

The busy period is the time between an arrival to an empty system and the epoch when the system becomes empty again. It is the first passage time T_{10}. The LT of the busy cycle (busy period plus subsequent independent idle period) is given in (2) on p. 210 of Takács [34], from which the transform of the busy period itself is easily obtained. Let the arrival rate be λ and the service rate be 1. Let $b(t)$ be the probability density function (pdf) of the busy period and let $\hat{b}(s)$ its Laplace transform. From [34], we obtain

$$\begin{aligned} \hat{b}(s) &\equiv \int_0^\infty e^{-st} b(t) dt = \frac{\lambda+s}{\lambda} \\ &\quad - \left[\lambda e^{-\lambda} \int_0^\infty \exp(-sx + \lambda e^{-x}) dx\right]^{-1} \; . \end{aligned} \tag{5.1}$$

Computation directly with (5.1) is inconvenient because of the integral.

There is a quite substantial literature related to the M/G/∞ busy period associated with type II particle counters and coverage problems; e.g., see Hall [25]. In particular, Stadje [33] shows that the complementary cdf (ccdf) of an M/G/∞

busy period has the form

$$B^c(t) \equiv \int_t^\infty b(u)du = \lambda^{-1} \sum_{n=1}^\infty c^{*n}(t), \quad t \ge 0 \ , \qquad (5.2)$$

where $c^{*n}(t)$ is a pdf, the n-fold convolution of a pdf with

$$c(t) = \lambda H^c(t) C^c(t), \quad t \ge 0 \ , \qquad (5.3)$$

$H^c(t)$ is the service-time ccdf and $C^c(t)$ is the ccdf of $c(t)$, which has the form

$$C^c(t) = \exp\left\{-\lambda \int_0^t H^c(u)du\right\} \ . \qquad (5.4)$$

Thus, in the M/M/∞ case,

$$c(t) = \lambda e^{-\lambda} \exp\{-t + \lambda e^{-t}\}, \quad t \ge 0 \ . \qquad (5.5)$$

While (5.2) and (5.5) provide interesting structural information, they do not seem so useful for computation.

In contrast, Dupuis and Guillemin [16] show that (5.1) can be given a series representation which is numerically useful, see p. 61 of [16], in particular,

$$\hat{b}(s) = \frac{\lambda + s}{\lambda} - \frac{e^\lambda s}{\lambda(1 + s\hat{\alpha}(s))} \ , \qquad (5.6)$$

where

$$\hat{\alpha}(s) = \sum_{n=1}^\infty \frac{\lambda^n}{(s+n)n!} \ . \qquad (5.7)$$

From (5.6) and (5.7), it is easy to get the first two moments

$$m_1 = (e^\lambda - 1)/\lambda \ \text{ and } \ m_2 = 2e^\lambda \sum_{n=1}^\infty \frac{\lambda^{n-1}}{n(n!)} \ . \qquad (5.8)$$

For example, for $\lambda = 1$ we get $m_1 = 1.72$, $m_2 = 7.17$ and $c^2 \equiv (m_2 - m_1^2)/m_1^2 = 1.42$. It is interesting that the squared coefficient of variation as a function of λ, $c^2(\lambda)$, approaches 1 both as $\lambda \to 0$ and as $\lambda \to \infty$. Moreover, $c^2(\lambda)$ first increases and then decreases, so that there is a λ maximizing $c^2(\lambda)$, in particular, $\lambda_{\max} \approx 2.97$ and $c^2(2.97) \simeq 1.73$. A rough exponential approximation based on matching the first two moments is

$$B^c(t) \approx \frac{2}{c^2+1} \exp(-2m_1 t/m_2) \ . \qquad (5.9)$$

Approximation (5.9) is supported by the fact that $B^c(t)$ is asymptotically exponential as $t \to \infty$; see p. 62 of Dupuis and Guillemin [16].

Dupuis and Guillemin also show that the busy-period pdf is a countably infinite mixture of exponentials. In addition, they determine the CF

$$\hat{b}(s) = -\frac{1}{\lambda} \Phi_{n=1}^\infty \frac{-\lambda n}{s + \lambda + n} \ , \qquad (5.10)$$

which we can also obtain as a special case of (4.9) by letting $\lambda_n = \lambda$ and $\mu_n = n$. Additional insight into the M/M/∞ transient behavior is provided by Preater [31].

The values of the Laplace transform $\hat{b}(s)$ are easily computed by either the series (5.7) or the CF (5.10). For the argument s needed in the numerical inversion, and for other representative s, we found that 30 terms of each was sufficient to produce 20-digit precision in the transform values. We then applied the Fourier-series method with Euler summation from Section 1 of [5] using parameters $A = 15 \log 10$, $m = n = 25$ to compute values of the ccdf $B^c(t)$. (Hence a sum of $m + n + 1 = 51$ terms had to be computed with s values $(A + 2k\pi i)/2t$, $k = 0, 1, \ldots, 50$.) Sample results are displayed in Table 2. The two different methods agreed at least to the 15 decimal places displayed.

From the last three entries of Table 2, we obtain an estimate of the exponential asymptotics for the tail probabilities

$$B^c(t) \sim 0.71200 \exp(-0.450265t) \quad \text{as} \quad t \to \infty \ . \qquad (5.11)$$

In contrast, approximation (5.9) yields $B^c(t) \approx 0.826e^{-0.480t}$.

In this case we did not actually need CFs, because the series representation is effective. We used the series representation to confirm the effectiveness of the CF. The situation is different if we consider the excursion time above some level c, i.e., the first-passage-time $T_{c+1,c}$. Guillemin and Simonian [24, p. 870] show that the Laplace transform of the excursion time in the M/M/∞ system can be represented as the ratio of two Kummer functions, i.e.,

t	$B^c(t)$
0.1	0.909, 084, 551, 819, 689
1.0	0.490, 128, 803, 420, 172
3.0	0.185, 450, 685, 115, 345
5.0	0.074, 977, 023, 964, 783
10.0	0.007, 888, 690, 353, 660
20.0	0.000, 087, 403, 419, 314
30.0	0.000, 000, 968, 394, 367
40.0	0.000, 000, 010, 729, 416
50.0	0.000, 000, 000, 118, 878

Table 2: Values of the M/M/∞ busy-period ccdf $B^c(t)$ computed by the Fourier-series method to 10^{-15} precision based on computing the LT values $\hat{b}(s)$ computed to 20-digit precision by both the series, (5.6) and (5.7), and the continued fraction (5.10).

$$\hat{f}_c(s) = \frac{c+1}{c+1+s} \frac{M(s, c+s+2, \lambda)}{M(s, c+s+1, \lambda)} \ , \qquad (5.12)$$

where M is the Kummer function

$$M(a, b, z) = \sum_{n=0}^\infty \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \ , \qquad (5.13)$$

with $(x)_n$ being the Pochhammer symbol, i.e., $(x)_0 = 1$ and $(x)_n = x(x+1)\ldots(x+n-1) = \Gamma(x+n)/\Gamma(x)$, where $\Gamma(x)$ is the gamma function. In this case, we know of no alternative for computation of $\hat{f}_{c+1}$ to the CF

$$\hat{f}_{c+1}(s) = -\frac{1}{\lambda} \Phi_{n=1}^\infty \frac{-\lambda(n+c)}{s + \lambda + n + c} \ , \qquad (5.14)$$

which again follows immediately from (4.9).

Giullemin and Simonian [24] also prove that the scaled excursion time $cT_{c+1,c}$ in the M/M/∞ system with arrival rate λc and individual service rate 1 converges to the busy period T_{10} in an M/M/1 system with arrival rate λ and service rate 1 as $c \to \infty$. We can establish additional results. We can make a stochastic comparison by noting that the scaled M/M/∞ system above level c is equivalent to a BD process with constant birth rate $\lambda_k = \lambda$ and death rates $\mu_k(c) = (c+k)/c$. Since $\mu_k(c)$ decreases to 1 as c increases, we can apply [36] again to conclude that the variables $cT_{c+1,c}$ increase stochastically in c, i.e.,

$$cT_{c+1,c} \leq_{st} (c+1)T_{c+2,c+1} \tag{5.15}$$

for all c, as well as converge in distribution as $c \to \infty$. Moreover, we can show that the entire scaled BD process above c converges to the M/M/1 queue-length process in the sense of weak convergence on function space. Similar observations are made by Preater [31].

The M/G/1 busy-period LT is known to satisfy the Kendall functional equation

$$\hat{b}(s) = \hat{g}(s + \lambda - \lambda\hat{b}(s)) \ , \tag{5.16}$$

where $\hat{g}(s)$ is the service-time LT. For M/M/1, $\hat{g}(s) = (1+s)^{-1}$, so that (5.16) becomes

$$\hat{b}(s) = (1 + s + \lambda - \lambda\hat{b}(s))^{-1} \ , \tag{5.17}$$

which upon iteration gives the CF

$$\hat{b}(s) = -\frac{1}{\lambda}\Phi_{n=1}^{\infty}\frac{-\lambda}{s+\lambda+1} \ , \tag{5.18}$$

as in (4.9). Of course, we can solve (5.17) to get

$$\hat{b}(s) = \frac{1}{2\lambda}\left(1 + \lambda + s - \sqrt{(1+\lambda+s)^2 - 4\lambda}\right) \ . \tag{5.19}$$

If we scale the M/M/1 busy-period pdf to have mean 1 and squared coefficient of variation $c^2 = v$ then the LT has the form

$$\hat{b}(s) = \frac{1}{v-1}\left(s + v - \sqrt{(s+v)^2 - (v^2-1)}\right) \ , \tag{5.20}$$

which can be written as a functional equation

$$\hat{b}(s) = \frac{(v+1)/2}{s + v - \left(\frac{v-1}{2}\right)\hat{b}(s)} \ . \tag{5.21}$$

From (5.21), we obtain the J fraction

$$\hat{b}(s) = \frac{-2}{v-1}\Phi_{n=1}^{\infty}\frac{-\frac{(v^2-1)}{4}}{s+v} \ . \tag{5.22}$$

However, from (2.26) on p. 161 of [2] and (94.23) on p. 376 of Wall [35], we see that the LT $\hat{b}$ has an S fraction of the form (3.10) with $a_1 = a_2 = 1$, $a_{2k-1} = (v-1)/2$ and $a_{2k} = (v+1)/2$ for all $k \geq 2$.

We now apply numerical inversion to evaluate the quality of the approximation for various values of c. Since the M/M/1 busy period pdf does not have a pure-exponential tail, whereas the M/M/∞ busy period has a pure exponential tail, we are led to expect some discrepancies for large values. The numerical results are displayed in Table 3. Consistent with our expectations, when t is small, the limiting M/M/1 values are good approximations when c is not large. However, as t increases, c needs to increase too in order for the approximation to be good.

time	Scaled M/M/∞			M/M/1
t	$c = 100$	$c = 1000$	$c = 10{,}000$	$c = \infty$
1	.48583276	.48930439	.48965265	.48969135
3	.24519851	.25005028	.25053923	.25059360
10	.08584378	.09192578	.09255093	.09262058
50	.00617291	.00983952	.01029354	.01034506
100	.00054713	.00173518	.00193941	.00196341
200	.00000571	.00010209	.00013611	.00014051
400	.00000000	.00000067	.00000143	.00000156

Table 3: The ccdf $B^c(t)$ for the scaled busy period ${}_cT_{c+1,c}$ in an M/M/∞ system with arrival rate c and service rate 1 for three values of c.

6. Beta Mixtures of Exponentials

In Abate and Whitt [8] we studied a class of pdf's obtained by taking beta mixtures of exponentials (BMEs), i.e.,

$$v(p,q;t) = \int_0^1 y^{-1}e^{-t/y}b(p,q;y)dy \ , \tag{6.1}$$

where $b(p,q;y)$ is the standard beta pdf, i.e.,

$$b(p,q,y) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}y^{p-1}(1-y)^{q-1}, \quad 0 \leq y \leq 1 \ , \tag{6.2}$$

$\Gamma(x)$ is the gamma function and $p > 0$ and $q > 0$. We observed that the BME pdf $v(p,q;t)$ has Laplace transform

$$\hat{v}(p,q;s) \equiv \int_0^\infty e^{-st}v(p,q;t)dt = {}_2F_1(1,p;p+q;-s) \ , \tag{6.3}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function.

In Section 2 of [8] and in [9], we showed that the BME pdf and its Laplace transform have Laguerre-series representations, which can be used for numerical calculations. For the BME LT, we obtain a closed-form expression for the Laguerre coefficients, so that

$$\hat{v}(p,q;s) = \frac{1}{1+s}\sum_{n=0}^{\infty}\frac{(q)_n}{(p+q)_n}\left(\frac{s}{1+s}\right)^n \ , \tag{6.4}$$

where $(x)_n$ is again the Pochhammer symbol. Here we observe that continued fractions tends to be more effective for this example.

The CF representation for the Gauss hypergeometric function in (6.3) was found by Gauss in 1812 by exploiting recursions; see p. 88 of Erdelyi [17]. The CF can also be constructed by the QD scheme mentioned at the end of Section 2 from the series representation of ${}_2F_1$; see p. 533 of Henrici [26]. For ${}_2F_1(1,p,p+q;-s)$, the elements can be taken to be $b_n = 1$, $n \ge 1$, $a_1 = 1$,

$$a_{2n} = \frac{(p+n-1)(p+q+n-2)s}{(p+q+2n-3)(p+q+2n-2)} , \quad n \ge 1 , \tag{6.5}$$
$$a_{2n+1} = \frac{n(q+n-1)s}{(p+q+2n-2)(p+q+2n-1)}, \quad n \ge 1 .$$

For the special case $p = 1/2$ and $q = 3/2$, the BME pdf coincides with the reflected Brownian motion (RBM) first-moment pdf, i.e., $\hat{h}_1(s) = \hat{v}(1/2, 3/2; s)$; e.g., see [8]. In that case, $a_{2n} = a_{2n+1} = 1/4$ for all $n \ge 1$. The special role of the RBM first-moment LT $\hat{h}_1(s)$ can be explained from the fact that it is the unique fixed point of the exponential mixture operator; see p. 93 of [6].

The CF associated with the BME transform tends to converge rapidly. This can be seen from the fact that $a_n \sim k/4$ (independent of n) as $n \to \infty$ in the S fraction for $\hat{v}(p,q;ks)$. Table 4 displays the first seven coefficients for the S fractions associated with the LTs of four distributions. One is an exponential; two are BME's; and the last is an exponential mixture of exponentials (EME) with pdf

$$f(t) = \int_0^\infty x^{-1} e^{-t/x} e^{-x} dx . \tag{6.6}$$

The exponential LT is a simple rational function, so it has a simple terminating CF. The two BME examples have $a_n = O(1)$ as $n \to \infty$, so the CF's are easily calculated. The EME has $a_n = O(n)$ as $N \to \infty$, so the CF can be calculated with some effort.

coef.	exp. $\frac{1}{1+s}$	BME $\hat{v}(1,1;2s)$	BME $\hat{v}(.5,1.5;4s)$	EME
a_1	1	1	1	1
a_2	1	1	1	1
a_3	0	1/3	1	1
a_4	0	2/3	1	2
a_5	0	2/5	1	2
a_6	0	3/5	1	3
a_7	0	3/7	1	3

Table 4: The coefficients of S fractions associated with Laplace transforms of four different pdf's.

To illustrate the computational advantages of continued fractions over the series representation (6.4), we consider the number of terms (in units of 10) required to compute the BME transform $\hat{v}(1,1;s)$ to obtain 16-digit precision. The numerical values for several s are displayed in Table 5.

As noted in Section 2 of [9], when we apply the Fourier-series method of numerical inversion of Laplace transforms incorporating Euler summation, we typically need to compute transform values at about 40 values of $s = u + iv$, with $u \approx 15/t$ and $v \approx k\pi/t$ for $k = 1, 2, \ldots, 40$. For the series in (6.4), the worst case (making $|s/(1+s)|$ close to 1) is $k = 40$. Then the required number of terms for the series (6.4) is approximately $n \approx 4.4t^{-1} \times 10^4$. In contrast, Table 5 shows that the continued-fraction method is much more efficient.

complex number s	series	continued fraction
1	50	30
3	130	40
5	190	50
9	330	60
15	540	70
20	710	80
25	890	90
$1 + 5i$	630	60
$2 + 10i$	1470	70
$15 + 125i$	44,000	250
$150 + 1250i$	440,000	710

Table 5: The number of terms needed to compute the BME transform $\hat{v}(1,1;s)$ to 16-digit precision by the series in (6.4) and the continued fraction in (1.7) and (6.5) for several values of s.

7. Distributions Associated with BMEs

Our ability to calculate BME LTs enables us to calculate LTs of several other important related distributions. First, in [8] we also considered a second beta mixture of exponentials, denoted by B_2ME, which is constructed by using the beta pdf of the second kind, i.e.,

$$v_2(p,q;t) \equiv \int_0^\infty y^{-1} e^{-t/y} b_2(p,q;y) dy, \quad t \ge 0 , \tag{7.1}$$

where

$$b_2(p,q;t) \equiv \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} y^{p-1} (1+y)^{-(p+q)}, \quad y \ge 0 . \tag{7.2}$$

Unlike the BME ccdf, the B_2ME ccdf has a long tail, i.e.,

$$V^c(p,q,t) \sim \frac{\Gamma(p+q)}{\Gamma(p)t^q} e^{-t} \text{ as } t \to \infty \tag{7.3}$$

and

$$V_2^c(p,q;t) \sim \frac{\Gamma(p+q)}{\Gamma(p)t^q} \text{ as } t \to \infty . \tag{7.4}$$

By Theorem 1.6 of [8], the ccdf's are related simply by

$$V_2^c(p,q;t) = e^t V^c(p,q;t), \quad t \ge 0 . \tag{7.5}$$

Thus, from (1.22) and (5.6) of [8], the LTs are related by

$$\hat{v}_2(p,q;s) = \frac{1}{s-1} (s\hat{v}(p,q,s-1) - 1) . \tag{7.6}$$

Thus, the CF for $\hat{v}(p,q,s-1)$ yields the LT $\hat{v}_2(p,q;s)$.

A third BME ccdf, denoted by B_3ME, is the dual ccdf as defined in Section 4, i.e.,

$$V_3^c(p,q,t) = \int_0^\infty e^{-tx} b(p,q;y)dy, \quad t \geq 0 \ , \tag{7.7}$$

for $b(p,q;y)$ in (6.2), with Laplace transform

$$\hat{V}_3^c(p,q;s) = s^{-1}\hat{v}(p,q;s^{-1}) \ . \tag{7.8}$$

The B_3ME LT is easily computed from the BME LT via (7.8).

In Section 7 of [8] we considered gamma mixtures of exponentials (GME) pdf's as limits of BME pdf's. Directly the GME pdf can be represented as

$$\begin{aligned} f(p;t) &= \int_0^\infty y^{-1}e^{-t/y}\frac{y^{p-1}e^{-y}}{\Gamma(p)}dy \\ &= \frac{2}{\Gamma(p)}t^{(p-1)/2}K_{p-1}(2\sqrt{t}) \ . \end{aligned} \tag{7.9}$$

In [8] we noted that the LT of $f(p;t)$ is

$$\hat{f}(p;s) = s^{-p}e^{1/s}\Gamma(1-p,s^{-1}) \ , \tag{7.10}$$

where $\Gamma(a,z)$ is the incomplete Gamma function.

In [8] we noted that $\hat{f}(p;s) = \lim_{q\to\infty} \hat{v}(p,q;qs)$. Hence, we can let $q \to \infty$ in the CF for ${}_2F_1(1,p;p+q;-qs)$ in (6.5) to obtain a CF representation of $f(p;s)$ of the form (3.10) with $a_{2n} = n+p-1$ and $a_{2n+1} = n$.

Now consider the dual ccdf associated with the pdf $f(p;t)$, obtained as a mean-1 exponential random variable divided by a gamma random variable, i.e.,

$$G^c(p;t) = \int_0^\infty e^{-t/y}\frac{e^{-1/y}}{y^{p+1}\Gamma(p)}dy = (1+t)^{-p} \ , \tag{7.11}$$

which is a Pareto distribution. Since $G^c(s) = s^{-1}\hat{f}(s^{-1})$, from (7.10) we obtain

$$\hat{G}^c(p;s) = s^{p-1}e^s\Gamma(1-p,s) \ , \tag{7.12}$$

which can be shown to be the Laplace transform of $(1+t)^{-p}$ from p. 21 of Oberhettinger and Badii [30]. Again the CF can be used to compute the transform values. By the duality, we see that $\hat{G}^c(p;s)$ has a CF of the form (3.11) with $a_{2n} = (n+p-1)$ and $a_{2n+1} = n$. Since the pdf $f(p;t)$ in (7.9) and ccdf $\hat{G}^c(t)$ in (7.11) are CM, it is natural to consider computations of the Laplace transforms by continued fractions.

8. Other Examples

We conclude with a few other examples of pdf's whose LTs can be effectively computed via CFs. These examples show that the CF elements can have remarkably simple structure.

Example 8.1. *First Bell pdf.* As in Example 6.1 of [9], consider the first Bell pdf with Laplace transform

$$\hat{f}(s) = \sum_{k=1}^\infty (e^{-1}/k!)(1+sk)^{-1} \ , \tag{8.1}$$

which has moments $m_n = n!b(n)$, where

$$b(n) \equiv e^{-1}\sum_{k=1}^\infty \frac{k^n}{k!} \tag{8.2}$$

is the n^{th} Bell number. By (8.1), the pdf is a countably infinite mixture of exponentials. The LT values can be effectively computed from the series (8.1), but they also can be effectively computed from CFs. The PD algorithm applied to these moments yields a CF of the form (2.8) with $a_{2n+1} = n$ and $a_{2n} = 1$. Since $a_{2n+1} = O(n)$, computation of the CF is possible but not easy.

Example 8.2. *Second Bell pdf.* As in Example 6.2 of [9], consider the second Bell pdf with Laplace transform

$$\hat{f}(s) = \sum_{k=1}^\infty 2^{-(k+1)}(1+sk)^{-1} \ , \tag{8.3}$$

which has moments $m_n = n!\tilde{b}(n)$, where

$$\tilde{b}(n) = \sum_{k=0}^\infty \frac{k^n}{2^{k+1}} \tag{8.4}$$

is the n^{th} ordered Bell number. This second Bell pdf is also a countably infinite mixture of exponentials. Again the LT values can be computed from either the series (8.3) or a CF. The PD algorithm applied to these moments yields a CF of the form (2.8) with $a_{2n+1} = 2n$ and $a_{2n} = n$. Again $a_n = O(n)$, so that computation of the CF is possible but not easy.

Example 8.3. EMIGs. As in [10] and Section 8 of [6], consider the two-parameter *exponential mixture of inverse Gaussian* (EMIG) pdf scaled to have mean 1 and squared coefficient of variation $c^2 > 1$. The pdf thus has explicit LT

$$\hat{f}(s) = \frac{c^2-1}{c^2-2+\sqrt{1+2(c^2-1)s}} \ . \tag{8.5}$$

From Theorem 4.1 of [10], the LT in (8.5) has integral representation

$$\hat{f}(s) = \tag{8.6}$$
$$\int_0^1 \frac{1}{1+2(c^2-1)sy}\frac{1}{\pi}\sqrt{\frac{1-y}{y}}\frac{c^2-1}{(1+[(c^2-2)^2-1]y)}dy \ .$$

By applying (94.22) on p. 375 of Wall [35], we see that the LT has the CF representation (3.10) with $a_1 = a_2 = 1$ and $a_k = (c^2-1)/2$ for all $k \geq 3$.

Since the LT is given explicitly in (8.5), the CF is not needed for computation. We display the CF because it has a remarkably simple form. For the special case $c^2 = 3$, $a_k = 1$ for all $k \geq 1$. The case $c^2 = 3$ corresponds to the RBM first-moment LT $\hat{h}_1(s)$. The CF for $\hat{h}_1(s)$ also can be obtained from the property that it is the unique fixed point of the exponential mixture operator, i.e.,

$$\hat{h}_1(s) = \frac{1}{1 + s\hat{h}_1(s)} \; ; \qquad (8.7)$$

see Section 7 of [6].

Example 8.4. *Weibull pdf's.* Consider a random variable X_r with the Weibull ccdf

$$F_r^c(t) = \exp(-r(t)^{1/r}), \quad t \geq 0 \; ; \qquad (8.8)$$

see Chapter 20 of Johnson and Kotz [27]. Since $X_r^{1/r}$ is distributed as X_1/r, the n^{th} moment of X_r is

$$m_n(X_r) \equiv E[X_r^n] = \frac{\Gamma(nr+1)}{r^{nr}} \; . \qquad (8.9)$$

Given the moments, we can construct the CF using the PD algorithm.

Here we are interested in the "long-tail" case of $r > 1$. The ccdf $F_r^c(t)$ in (8.8) is CM in that case, so that the CF of the LT is an S fraction. It is known that the moments do *not* determine the distribution in that case. The indeterminateness follows from the Krein condition, i.e., for $r > 1$, the divergence

$$\int_0^\infty (1+x^2)^{-1} \log f_r(x) dx = -\infty \qquad (8.10)$$

fails to hold, where f_r is the pdf associated with F_r; e.g. see Akhiezer [11].

For integer $r > 1$, we can identify products of random variables (mixtures of distributions) that have the moments of X_r. (We conjecture that the full distributions coincide with the Weibull as well.) For integer r,

$$\begin{aligned} m_n(X_r) &= \frac{(nr)!}{r^{nr}} = \left(\frac{1}{r}\right)_n \left(\frac{2}{r}\right)_n \cdots \left(\frac{r-1}{r}\right)_n (1)_n \\ &= \frac{(1/r)_n}{(1)_n} \frac{(2/r)_n}{(1)_n} \cdots \frac{((r-1)/r)_n}{(1)_n} (1)_n^r, \quad (8.11) \end{aligned}$$

where as before $(x)_n$ is the Pochhammer symbol. However, recall that for $0 < p \leq 1$, $0 < q < 1$, $(1)_n = n!$, $(p)_n$ and $(p)_n/(p+q)_n$ are the moment sequences of the exponential, gamma and beta pdf's; e.g., see [8]. Let $Z(p)$ be a random variable with a gamma pdf having shape parameter p, as in (7.9), so that $Z(1)$ has a mean-1 exponential pdf, and let $Y(p,q)$ be a random variable having the beta pdf in (6.3) with parameters p and q. Then, for integer $r > 1$,

$$\begin{aligned} E[X_r^n] &= E \prod_{k=1}^{r} Z(k/r) \\ &= E\left[\prod_{k=1}^{r-1} Y(k/r,(r-k)/r) \prod_{k=1}^{r} Z_k(1)\right] \quad (8.12) \end{aligned}$$

for all n, where the random variables on the right are mutually independent, with $Z_k(1)$ all distributed as $Z(1)$.

For the case $r = 2$, we can conclude that they have the same distributions, because $F_r^c(t)$ is CM for $r > 1$. Hence,

$$X_2 \stackrel{\text{d}}{=} Z(1/2)Z(1) \; , \qquad (8.13)$$

where $\stackrel{\text{d}}{=}$ denotes equality in distribution. Since the gamma (1/2) pdf of $Z(1/2)$ is determined by its moments and X_2 is CM, (8.13) is justified. Moreover, we can apply Theorem 3.1 to conclude that the S fraction associated with the LT Ee^{-sX_2} converges.

However, for integer $r > 2$, we can conclude that the S fraction associated with Ee^{-sX_r} fails to converge, because the moments $m_n(X_r)/n!$ do *not* determine the mixing cdf H.

To illustrate, the first 7 numerator elements of the CF (3.10) for Ee^{-sX_r} are given in Table 6. (In Table 6 all distributions are scaled to have mean 1.) Since $a_n = n-1$ for X_2, we see that convergence takes place, but it is not too rapid. For $r = 3$ and 4, we see that a_n fails to be $O(n)$ as $n \to \infty$. Moreover, for $r = 3, 4$ the CF fails to converge, demonstrating that the moments of the mixing cdf H indeed do not determine H.

	continued fraction elements						
r	a_1	a_2	a_3	a_4	a_5	a_6	a_7
2	1	1	2	3	4	5	6
3	1	1	9	20	40	61.25	93.15
4	1	1	34	133.82	364.25	736.02	1342.57

Table 6: The first seven CF numerator elements, for a CF of the form (3.10), of Weibull Laplace transforms for $r = 2, 3$ and 4.

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