IEOR 3106: Introduction to Operations Research: Stochastic Models

SOLUTIONS to First Midterm Exam, Part II, October 12 in class

Open Book: but only the Ross textbook plus one 8×11 page of notes

Justify your answers; show your work.

1. Dry and Rainy Days. (12 **points)**

Suppose that the weather has been classified as being either dry or rainy. Then it is observed that: "One out of two rainy days is followed by a dry day, but only one out of four dry days is followed by a rainy day."

(a) Construct a Markov chain model based on that observation.

Please note that this is a minor variation of the "cars-and-trucks" problem - Homework Exercise 4.30, with the state names changed in order to confuse and befuddle you. Hopefully you saw through this subterfuge. As usual, the Markov chain model is defined by defining the Markov chain transition matrix P. Here the transition matrix should be:

$$P = \begin{matrix} D \\ R \end{matrix} \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix} .$$

The columns are understood to be labelled in the same way as the rows.

Of course, you could have ordered the states differently, and obtained instead

$$P = \begin{matrix} R \\ D \end{matrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix} .$$

You would get the same answers for the rest of the questions.

(b) According to that model, given that it is rainy one day, what is the probability that the next three days will also be rainy?

We want $(P_{R,R})^3 = (1/2)^3 = 1/8$.

(c) According to that model, what is the long-run proportion of rainy days?

We should solve $\pi = \pi P$ with $\pi_1 + \pi_2 = 1$. That reduces to $(x, 1-x) = (x, 1-x)P$, which reduces to the single equation

$$x = \frac{3x}{4} + \frac{(1-x)1}{2} ,$$

which has the solution $x = 2/3$. Hence $\pi = (2/3, 1/3)$. You should verify that this works: To avoid arithmetic errors, check that $\pi = \pi P$ with this π. Finally the long run proportion of rainy days is $\pi_R = 1/3$. (If you had ordered the states in the other order, then you would get $\pi = (1/3, 2/3)$ but still $\pi_R = 1/3$).

(d) Does the original observation (even if accurate) imply that a Markov chain model is necessarily appropriate? Explain.

No, as discussed at length in class, the original statement does not directly imply the Markov property. In class we demonstrated that by presenting an alternative deterministic model, which is consistent with the statement, and yet does not satisfy the Markov property. The idea is to present a deterministic repetitive pattern. For this problem, the pattern should be (R, R, D, D, D, D). This pattern is repeated forever; i.e., the successive days can be:

$$R, R, D, D, D, D, R, R, D, D, D, D, R, R, D, D, D, D, R, R, D, D, D, D, R, R, D, D, D, D, \ldots$$

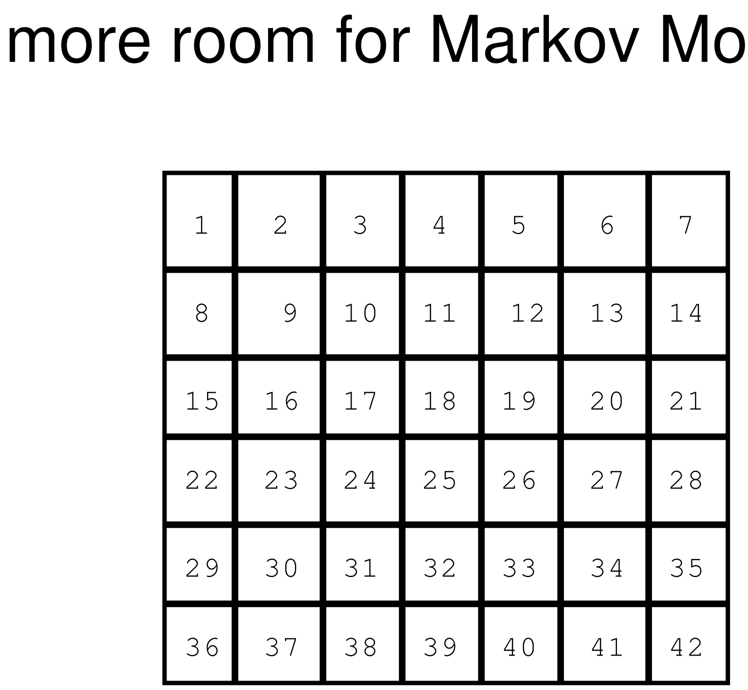
Note that the pattern of length 6 is repeated indefinitely. Note that 1 of 2 rainy days (R) is followed by a dry day (D), but only 1 out of 4 dry days is followed by a rainy day. (The last D in each pattern of length 6 is followed by the R in the first place in the next pattern.)

To see that this is a different model, note that the answer to part (b) would become very different. And yet the long-run proportion of rainy days is actually correct by the Markov chain model. That is true in general, but we have not proved it.

---

2. more room for Markov mouse (18 points)

Once again (as in class), we consider Markov mouse moving around in a closed maze, but now we give him 42 rooms instead of only 9, as shown in the figure below.



As before, Markov mouse moves randomly from room to room through doors (vertically and horizontally) connecting neighboring rooms, with his next move being independent of the history leading up to his current room. In particular, he chooses each of the neighboring rooms at each move with equal probability. For example, from Room 8, Markov mouse can move to Room 1, Room 9 or Room 15 in his next move, and does each one of these with probability 1/3. On the other hand, from Room 7, he can move next to either Room 6 or Room 14, and selects each of these with probability 1/2.

(a) Can this problem be modelled as a Markov chain and, if so, how many states does the Markov chain have?

Yes, this problem can be modelled as a Markov chain. It has 42 states, one for each room. (This is a "freebie."

(b) Is this Markov chain irreducible and aperiodic? Explain.

The DTMC is irreducible; it is possible to go from each state to any other state in some finite number of steps. However, the DTMC is periodic, just like the 9-state example discussed in class. From an even state, Markov mouse (MM) most go next to an odd state; from an odd state, MM must go to an even state. The DTMC has period 2.

(c) Let X_n be the room occupied by Markov mouse after making n moves. Does $P(X_n = 23|X_0 = 1)$ converge to a limit as $n \to \infty$ and, if so, what is the limit?

No, because of the periodicity, these probabilities oscillate between positive numbers and 0.

(d) What is the long-run proportion of moves (transitions) in which Markov mouse *ends up in* Room 13?

Now we exploit the time reversibility, as in Section 4.8 of Ross. We know that this DTMC is equivalent to a random walk on a weighted graph. The nodes of the graph are the rooms. The edges are the doorways to neighboring rooms. In this example, all the edge weights are 1. For any room (state) j,

$$\pi_j = \frac{\text{number of doors out of room } j}{\text{sum over all rooms of the number of doors out of the room}} \ .$$

Note that the denominator is the sum of all the doors *multiplied times* 2, because the door should be counted twice, once for each of the two rooms it connects.

Here there are $5 \times 4 = 20$ rooms with 4 doors each, yielding 80 doors. Then there are $4 + 5 + 4 + 5 = 18$ rooms with 3 doors each, yielding 54 doors. Finally, there are the 4 corner rooms with 2 doors each, yielding 8 doors. So the denominator should be $80 + 54 + 8 + 142$ doors.

Since Room 13 has 4 doors, we have

$$\pi_{13} = \frac{4}{142} = \frac{2}{71} \ .$$

Note that 4/71 is the *wrong* answer, because that would be the total number of doors, not counting each one twice (once for each room it connects).

We remark that $P^{100}_{13,13} \approx 2\pi_{13} = 4/71$, while $P^{101}_{13,13} = 0$, so that the long-run proportion is 2/71, which is approximately $(P^{100}_{13,13} + P^{101}_{13,13} = 0)/2$.

(e) What is the long-run proportion of all moves Markov mouse makes that take him from Room 23 to Room 24?

The question should be read carefully. In general, the answer is $\pi_{23} P_{23,24}$. For this example, we have $\pi_{23} P_{23,24} = (4/142)(1/4) = 1/142$.

(f) Starting in Room 25, what is the expected number of moves until Markov mouse first returns to Room 25?

In general, the answer is $1/\pi_{25}$; see Remark (ii) on page 208 of Ross. Here, then, the answer is $1/\pi_{25} = 142/4 = 71/2 = 35.5$

---

3. A 10×10 Markov chain transition matrix (20 points)

Consider a Markov chain on the ten states $\{1, 2, \cdots, 10\}$ with transition matrix P given by

$$P = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{matrix} \begin{pmatrix} 0.3 & 0.1 & 0.0 & 0.0 & 0.0 & 0.2 & 0.1 & 0.1 & 0.2 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.2 & 0.0 & 0.0 & 0.8 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.0 & 0.0 & 0.0 & 0.4 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.8 & 0.0 & 0.0 & 0.2 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 \\ 0.1 & 0.1 & 0.0 & 0.0 & 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.1 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.2 & 0.0 & 0.8 & 0.0 & 0.0 & 0.0 \end{pmatrix}$$

**Note that we are numbering the states 1, 2, ... , 10, with the columns numbered in the same order as the rows.

(a) Which states are accessible from state 1?

Remember that j being *accessible* from i means that you can get from i to j in some finite number of steps, not necessarily in a single step. It turns out that *all* 10 states are accessible from state 1. This can be determined by constructing a graph showing the $1-$*step* connectivity.

It becomes clear from the canonical form of the transition matrix, in part (g). To answer this part, you should be beginning your construction of the canonical form.

(b) From which states is state 1 accessible?

The principle is the same, but it is easier to leave state 1 than to get to it. In fact, state 1 is accessible only from states 1 and 9.

(c) Do states 1 and 6 communicate?

States 1 and 6 do *not* communicate. They communicate if each is accessible from the other. That is not true.

(d) Identify the communication classes for this Markov chain.

Each communication class is a subset of the states. When we form communication classes, we construct a *partition* of the set of states (into disjoint subsets whose union is the whole set). Here there are 5 communication classes: $\{2\}$, $\{3,6\}$, $\{4,8\}$, $\{5,7,10\}$, and $\{1,9\}$.

(e) Which communication classes are closed? Which are open?

The closed classes are the classes that you cannot leave. The open classes are the classes from which you can leave, and thus eventually will leave with probability 1. There are four closed classes: $\{2\}$, $\{3,6\}$, $\{4,8\}$ and $\{5,7,10\}$. There is one open class: $\{1,9\}$.

(f) Which states are transient? Which states are recurrent?

The states in closed communication classes are recurrent: You will return with probability 1. The states in open communication classes are transient: You will eventually leave for the last time and never return again after that. The transient states are 1 and 9; the others are recurrent.

(g) Put the transition matrix in canonical form.

Here is the canonical form of the transition matrix P:

$$
P = \begin{array}{c} 2 \\ 3 \\ 6 \\ 4 \\ 8 \\ 5 \\ 7 \\ 10 \\ 1 \\ 9 \end{array}
\left(\begin{array}{cccccccccc}
1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.2 & 0.8 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.8 & 0.2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.6 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.2 & 0.8 & 0.0 & 0.0 & 0.0 \\
0.1 & 0.0 & 0.2 & 0.0 & 0.1 & 0.0 & 0.1 & 0.0 & 0.3 & 0.2 \\
0.1 & 0.0 & 0.2 & 0.0 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1
\end{array} \right)
$$

We have re-ordered the states so that the 4 closed communication classes appear together at the top, while the single open communication class appears together at the bottom. The original states are lasted on the left. As before, the columns and rows both have this ordering. For example, the entry 1 in the upper left corner is the transition probability $P_{2,2}$, because state 2 has been moved to being first. We have ordered the four closed communication classes by size, putting the smaller ones first, but that is optional. You could have a different matrix, but the states in the same communication class must appear together, next to each other, and the transient states must appear at the bottom.

In the following questions, we are referring to the states as originally defined and numbered.

(h) Compute the six-step transition probability $P^{(6)}_{2,7}$.

Note that state 7 is not accessible from state 2, so this is another "freebie." Here $P^{(6)}_{2,7} = 0$.

(i) Compute the two-step transition probability $P^{(2)}_{4,8}$.

Note that states 4 and 8 belong to the same communicating class, so here there is something to compute. It suffices to look at the little 2×2 sub-matrix, which is a transition matrix in its own right. We have

$$
P = \begin{array}{c} 4 \\ 8 \end{array} \left(\begin{array}{cc} 0.6 & 0.4 \\ 0.5 & 0.5 \end{array} \right) .
$$

The columns are understood to be labelled in the same way as the rows. To proceed, we calculate the two-step transition matrix $P^{(2)} = P^2$ for this little DTMC. We get

$$
P^2 = \begin{array}{c} 4 \\ 8 \end{array} \left(\begin{array}{cc} 0.56 & 0.44 \\ 0.55 & 0.45 \end{array} \right) .
$$

Hence, $P^{(2)}_{4,8} = 0.44$.

(j) Compute the two-step transition probability $P^{(2)}_{1,2}$.

Things are a little more complicated here, because 1 is a transient state, while 2 is an absorbing state. But you just need to think clearly: There are *three* possibilities, i.e.,

$$P^{(2)}_{1,2} = P_{1,2}P_{2,2} + P_{1,1}P_{2,2} + P_{1,6}P_{6,2} = (0.1)(1.0) + (0.3)(0.1) + (0.2)(0.1) = 0.15$$

(k) Starting in state 3, what is the expected total number of visits to state 3?

State 3 is a recurrent state, belonging to a closed communicating class. Thus the expected total number of visits to state 3, starting in state 3, is infinite.

(l) Starting in state 1, what is the expected total number of visits to state 5?

This might at first look complicated, but you only need remember that infinity times any positive number is infinity, and infinity plus any finite number is infinity. There is a positive probability of going from state 1 to state 10 in one step; once in state 10, the expected number of visits to state 5 is infinite. So the expected number of visits to state 5 starting in state 1 is infinite. This is true even though we cannot get from state 1 to state 5 in one step, and there is a significant chance (probability) that we will *never* reach state 5 from state 1.

(m) Starting in state 3, what is the expected number of steps before you return again to state 3?

State 3 is in a closed communicating class (with state 6), so the answer here is the reciprocal of the steady-state probability, i.e., $1/\pi_3 = 2.0$; see Remark (ii) on page 208 of Ross; see Chapter 7 for more on this point. The steady-state probability itself is immediate because the 2×2 Markov chain transition matrix involving state 3 is doubly stochastic (columns sums are 1); see Problem 4.20 in Ross.

(n) Starting in state 1, what is the expected total number of visits to state 9?

Finally, we have come to a question that requires some work. These last two parts require that you know how to work with the fundamental matrix of an absorbing chain $N = (I - Q)^{-1}$. Even though the entire chain is not an absorbing chain, we can analyze movement among the two transient states as if it were. Here then we focus on the sub-matrix Q corresponding to thew two transient states 1 and 9. We have

$$Q = \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 0.3 & 0.2 \\ 0.1 & 0.1 \end{pmatrix} .$$

The columns are understood to be labelled in the same way as the rows. Then we have

$$I - Q = \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 0.7 & -0.2 \\ -0.1 & 0.9 \end{pmatrix}$$

Now, for the first time, you have to do some mathematics; you need to invert a 2×2 matrix, but even if you cannot do that, perhaps because you ran out of time, you could at least quickly write own the formula. You want $N_{1,9}$ where $N = (I - Q)^{-1}$. All that is left is the linear algebra: The answer is

$$N = (I - Q)^{-1} = \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 90/61 & 20/61 \\ 10/61 & 70/61 \end{pmatrix} .$$

You could easily mess up in this step, but it is easy to check your answer; you should have $N \times (I - Q) = I$. That is easy to check. Because of the probabilistic interpretation (and the associated mathematical structure), the elements of N must all be nonnegative. Here is the gruesome analysis in detail: The successive operations on the matrix $I - Q$ look like this:

$$I - Q = \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 0.7 & -0.2 \\ -0.1 & 0.9 \end{pmatrix}$$

$$\text{step 1:} \quad \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 1 & -2/7 \\ -0.1 & 0.9 \end{pmatrix}$$

$$\text{step 2:} \quad \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 1 & -2/7 \\ 0 & 61/70 \end{pmatrix}$$

$$\text{step 3:} \quad \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 1 & -2/7 \\ 0 & 1 \end{pmatrix}$$

$$\text{step 4:} \; I = \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The corresponding operations applied to the identity matrix are:

$$I = \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{step 1:} \quad \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 10/7 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{step 2:} \quad \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 10/7 & 0 \\ 1/7 & 1 \end{pmatrix}$$

$$\text{step 3:} \quad \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 10/7 & 0 \\ 10/61 & 70/61 \end{pmatrix}$$

$$\text{step 4:} \; (I - Q)^{-1} = \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 90/61 & 20/61 \\ 10/61 & 70/61 \end{pmatrix}$$

(o) What is the approximate value of $P_{1,6}^{(25)}$?

And here is yet another question that requires some work, but not too much if you know what you are doing. Note that state 1 is a transient state, while 6 is one of the recurrent states that we might get to from state 1. We need to compute the probability of being absorbed in the closed communicating class containing state 6, starting from state 1. Then we need to multiply by the stationary probability of being in state 6 for the irreducible aperiodic two-state

DTMC containing states 6 and 3. We first collapse the closed communicating classes to single states in order to construct an absorbing Markov chain. We replace the closed communicating classes by single states. For the transitions from transient to communicating classes, we must appropriately add the values.

The new reduced transition matrix has 6 states: one for each of the 4 closed communicating classes and 2 for the 2 transient states. The new reduced absorbing transition matrix is:

$$P = \begin{matrix} 2 \\ \{3,6\} \\ \{4,8\} \\ \{5,7,10\} \\ 1 \\ 9 \end{matrix} \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.1 & 0.1 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.3 & 0.1 & 0.1 \end{pmatrix}$$

As discussed in the lecture of September 28, here the absorbing DTMC has the general form:

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} ,$$

where I is an identity matrix (1's on the diagonal and 0's elsewhere) and 0 (zero) is a matrix of zeros. In this case, I would be 4×4, R is 2×4 and Q is 2×2. The matrix Q describes the probabilities of motion among the transient states, as already discussed. The matrix R gives the probabilities of absorption in one step (going from one of the transient states to one of the absorbing states in a single step). Here the absorbing states correspond to entire communicating classes in the original DTMC with 10 states. In general Q would be square, say m by m, while R would be m by k, and I would be k by k.

In this framework, we want to compute the matrix $B = NR$, using the fundamental matrix N already computed in the previous part (n). Since 1 is the 5^{th} state in this new matrix and since the target destination state 6 belongs to the second communication class, we want to compute $B_{5,2}$. We then want to multiply that by π_6, obtained from the original 2×2 DTMC containing for state 6.

Here

$$B = NR = \begin{matrix} 1 \\ 9 \end{matrix} \begin{pmatrix} 11/61 & 22/61 & 13/61 & 15/61 \\ 8/61 & 16/61 & 15/61 & 22/61 \end{pmatrix}$$

As a check, note that the entries must be probabilities. The entries are nonnegative and the row sums are 1. We want $B_{1,2} = 22/61$. (State 5 in the 6-state chain became state 1 as labelled.)

To find π_6, we must solve $\pi = \pi P$ for the 2×2 irreducible DTMC

$$P = \begin{matrix} 3 \\ 6 \end{matrix} \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix}$$

Our task is easy, though, because the column sums are 1, i.e., the DTMC is doubly stochastic (as in Exercises 4.20-4.22), so that $\pi = (1/n, 1/n) = (1/2, 1/2)$. So $\pi_6 = 1/2$.

Hence the desired answer is $(B_{5,2} \equiv B_{1,2}) \times \pi_6 = (22/61) \times (1/2) = 11/61$. I apologize for the confusion because of the changing state labels. We want to make sure in the end that what we are calculating applies to the original states 1 and 6. State 6 changed to state 2 in the new absorbing transition matrix. State 1 changed into state 5 in that absorbing matrix, but then state 1 in the @ $\times$ 4 matrices R and B that we finally compute. We make sure that we compute what we really want to compute.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing "I have neither given not received improper help on this examination," on your examination booklet and sign your name. You may keep the exam itself. Solutions will eventually be posted on line.