IEOR 3106: Introduction to Operations Research: Stochastic Models
Part II of First Midterm Exam, Ch. 4, Tuesday, October 9, 2007

Justify your answers; show your work.

1. The Cheerful Professor. (12 points)

The professor’s mood can either be cheerful or gloomy. It has been observed that: “One out of two gloomy days is followed by a cheerful day, but only one out of nine cheerful days is followed by a gloomy day.”

(a) Construct a Markov chain model based on that observation.

(b) According to that model, given that the professor is gloomy one day, what is the probability that he is gloomy on each of the next three days?

(c) According to that model, what is the long-run proportion of days the professor is gloomy?

(d) Does the original observation (even if accurate) imply that a Markov chain model is appropriate? Explain.

2. A Strange Random Clock. (23 points)

In this problem we consider a strange clock, which has only an hour hand. The hour hand moves at most once per hour, on the hour. At each hour, the hour hand makes a random move to one of the 12 hours, with the new hour depending only on the current position of the hour hand. Specifically, if the current hour (on the clock) is \( k \), \( 1 \leq k \leq 12 \), then the hour hand moves clockwise (around to the right) \( k \) hours ahead, with probability \( 1/2 \). But, also with probability \( 1/2 \), the hour hand remains unchanged at hour \( k \). For example, from 10, the hour hand can next move to 10 or 8, and each occurs with probability \( 1/2 \). (Then 8 is 10 hours clockwise to the right from hour 10.) Let \( X_n \) be the time on the clock (position of the hour hand) after the \( n \)th hour. Let \( X_0 \) be the initial position of the hour hand.

(a) Construct a Markov chain model of the successive times shown (positions of the hour hand) on this clock at successive hours.

(b) What states are accessible from state 4?

(c) Do states 2 and 6 communicate?

(d) What is \( P(X_{n+7} = 12|X_n = 12) \)?

(e) What is \( P(X_{n+7} = 8|X_n = 4) \)?

(f) What is \( P(X_{n+3} = 10|X_n = 5) \)?

(g) Construct the canonical form of the probability transition matrix of this DTMC.

(h) Is this Markov chain irreducible? Explain.

(i) Is this Markov chain periodic? Explain.

(j) What are the transient states of this DTMC?

(k) What are the closed communication classes of this DTMC?

(l) Suppose that \( P(X_0 = k) = 1/12 \) for \( 1 \leq k \leq 12 \). Find the approximate probability distribution of \( X_n \) for large \( n \).
(m) Does there exist a probability vector \( \pi \) satisfying the equation \( \pi = \pi P \)? And, if so, is it unique (among probability vectors of that length)? Explain.

3. Random Walk on a Graph. (15 points)

Consider a random walk on the graph containing 8 nodes and 10 edges shown in the figure below. There is a weight on each arc or edge. Suppose that we move from node to node in each step, going from each node only to a neighboring node (connected by an arc). Let the neighboring node to which the walk goes be selected at random with probability proportional to the weight on the arc to that node. For example, \( P_{G,F} = 1/4 \), while \( P_{G,H} = 3/4 \).

(a) Starting in node \( C \), what is the probability of being in node \( C \) again after three steps? (You need not do the arithmetic to simplify the answer.)

(b) Is the Markov chain irreducible?

(c) Is the Markov chain periodic? If so, what is the period?

(d) What is the long-run proportion of steps that the random walk spends in node \( E \)?

(e) What is the expected number of steps, starting in node \( E \), until the random walk next visits node \( E \)?

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