IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2007, Professor Whitt, Final Exam

Chapters 4-7 and 10 in Ross, Tuesday, December 18, 4:10pm-7:00pm

Open Book: but only the Ross textbook, the CTMC Notes, and three 8×11 pages of notes

Honor Code: Students are expected to behave honorably when taking exams. After completing the test, please testify to your adherence by writing the following on your bluebook and signing your name: "I have neither given nor received help while taking this test."

Justify your answers; show your work.

1. **A king moving randomly on a chessboard** (30 points)

A king is a chess piece that can move one square in any direction, horizontally (along a row), vertically (along a column) or diagonally. For example, from a square in the middle of the board, the king has 8 possible legal moves, but from a corner square, the king has only 3 possible legal moves. Suppose that a king is initially placed on a specified corner square of an empty 8×8 chessboard. Suppose that the king makes a sequence of independent random moves, with each of its available legal moves chosen with equal probability on each move.

(a) Let the 64 squares be numbered. Let X_n be the square occupied by the king after n moves. Indicate whether or not each of the following statements is TRUE or FALSE:

1. The stochastic process $\{X_n : n \geq 0\}$ is a (discrete-time) renewal counting process.
2. The stochastic process $\{X_n : n \geq 0\}$ is a (discrete-time) Markov chain.
3. The stochastic process $\{X_n : n \geq 0\}$ is an irreducible Markov chain.
4. The stochastic process $\{X_n : n \geq 0\}$ is a periodic Markov chain.
5. The stochastic process $\{X_n : n \geq 0\}$ is a recurrent Markov chain.
6. The stochastic process $\{X_n : n \geq 0\}$ is an absorbing Markov chain.

(b) Let $N(n)$ be the number of times that the king visits its initial square in the first n moves, with $N(0) \equiv 0$. Indicate whether or not each of the following statements is TRUE or FALSE:

1. The stochastic process $\{N(n) : n \geq 0\}$ is a Poisson process.
2. The stochastic process $\{N(n) : n \geq 0\}$ is a (discrete-time) renewal counting process.
3. The stochastic process $\{N(n) : n \geq 0\}$ is a Markov chain.
4. The stochastic process $\{N(n) : n \geq 0\}$ is a periodic Markov chain.
5. The stochastic process $\{N(n) : n \geq 0\}$ has independent increments.
6. $E[N(n)]/n$ necessarily converges to a positive limit as $n \to \infty$.

(c) What is the probability that the king is again at its initial square after three moves?

(d) What is the long-run proportion of moves that the king is at its initial square?

(e) What is the expected number of moves until the king first returns to its initial square?

(f) Give an expression (formula or formulas) for the expected number of moves until the king first visits the opposite corner square?

2. Back and Forth to Campus (20 **points)**

Professor Prhab Hubilliti lives at the bottom of the hill on the corner of 117^{th} Street and 7^{th} Avenue. Going each way - up hill to to teach his class at Columbia or down hill back home - Prhab either runs or walks. Going up the hill, Prhab either walks at 2 miles per hour or runs at 5 miles per hour. Going down the hill, Prhab either walks at 3 miles per hour or runs at 6 miles per hour. In each direction, he always runs the entire way or walks the entire way. Since Prhab often works late into the night, he often gets up late, and has to run up hill to get to his class. On any given day, Prhab runs up hill with probability 2/3 and walks up hill with probability 1/3. On the other hand, Prab is less likely to run going back home. On any given day, he runs down hill with probability 1/3 and walks down hill with probability 2/3.

(a) What is the average speed Prhab goes up the hill to campus? What is the average speed Prhab goes down the hill back home?

(b) What is the long-run proportion of Prhab's total travel time going to and from campus that he spends going up hill to campus?

(c) What is the long-run proportion of Prhab's total travel time going to and from campus that he spends running up hill to campus?

3. **The Boberg-Kapoor Investment Company** (25 points)

Ross Boberg and Arjun Kapoor have been so successful with their Boberg-Kapoor Investment Company that their company has gone public and is now listed on the New York Stock Exchange under the BK symbol. Suppose that a share of BK stock initially cost \$100. Suppose that the BK share price over time (measured in years) is modelled as the stochastic process

$$S(t) \equiv 100 + 10B(t),$$

where $\{B(t) : t \geq 0\}$ is standard Brownian motion, having mean $E[B(t)] = 0$ and variance $Var(B(t)) = t$ for $t \geq 0$.

(a) What is the probability that the BK share price exceeds \$120 after 4 years?

(b) What is the probability that the BK share price will exceed \$120 any time during the first 4 years?

(c) What is the probability that the BK share price rises to \$140 before it drops to \$80, and then after hitting \$140, rises further to \$180 before it drops back to \$120?

(d) What is the expected time until the BK share price first either rises to \$140 or drops to \$80?

(e) What is the conditional expectation:

$$E[S(2)S(3)|S(1) = 100]?$$

4. **The DDT Barbershop** (25 points)

Nishi Dedania, Rahul Dhir and Arita Thatte have joined together to form the DDT barbershop. The DDT barbershop has room for at most five customers, with up to three in service and two waiting. Suppose that potential customers arrive according to a Poisson process at a rate of 8 per hour. Suppose that potential arrivals finding the barber shop full, with three customers in service and two other customers waiting, will leave and not affect future arrivals. Suppose that successive service times are independent exponential random variables with mean 15 minutes. Suppose that waiting customers have limited patience, with each waiting customer being willing to wait only an independent random, exponentially distributed, time with mean 15 minutes before starting service; if the customer has not started service by that time, then the customer will abandon, leaving without receiving service.

(a) What proportion of time are all three barbers busy in the long run?

(b) What is the total rate of customer abandonment in the long run?

(c) What proportion of all potential arrivals are served?

(d) What is the probability that, starting empty, the third customer arrives before anybody has completed service?

(e) What is the probability that, starting empty, the fourth customer arrives and abandons before anybody has completed service?