

IEOR 3106: Introduction to Operations Research: Stochastic Models
Fall 2007, Professor Whitt, Final Exam, SOLUTIONS

Chapters 4-7 and 10 in Ross, Tuesday, December 18, 4:10pm-7:00pm

Open Book: but only the Ross textbook, the CTMC Notes, and three
 8×11 pages of notes

Honor Code: Students are expected to behave honorably when taking exams. After completing the test, please testify to your adherence by writing the following on your bluebook and signing your name: "I have neither given nor received help while taking this test."

Justify your answers; show your work.

1. A king moving randomly on a chessboard (30 points)

A king is a chess piece that can move one square in any direction, horizontally (along a row), vertically (along a column) or diagonally. For example, from a square in the middle of the board, the king has 8 possible legal moves, but from a corner square, the king has only 3 possible legal moves. Suppose that a king is initially placed on a specified corner square of an empty 8×8 chessboard. Suppose that the king makes a sequence of independent random moves, with each of its available legal moves chosen with equal probability on each move.

(a) Let the 64 squares be numbered. Let X_n be the square occupied by the king after n moves. Indicate whether or not each of the following statements is TRUE or FALSE:

1. The stochastic process $\{X_n : n \geq 0\}$ is a (discrete-time) renewal counting process. **F**
2. The stochastic process $\{X_n : n \geq 0\}$ is a (discrete-time) Markov chain. **T**
3. The stochastic process $\{X_n : n \geq 0\}$ is an irreducible Markov chain. **T**
4. The stochastic process $\{X_n : n \geq 0\}$ is a periodic Markov chain. **F**
5. The stochastic process $\{X_n : n \geq 0\}$ is a recurrent Markov chain. **T**
6. The stochastic process $\{X_n : n \geq 0\}$ is an absorbing Markov chain. **F**

Scoring: 0 wrong: 5 points, 1 wrong: 3 points, 2 wrong: 0 points, ≥ 3 wrong: -1 point.

(b) Let $N(n)$ be the number of times that the king visits its initial square in the first n moves, with $N(0) \equiv 0$. Indicate whether or not each of the following statements is TRUE or FALSE:

1. The stochastic process $\{N(n) : n \geq 0\}$ is a Poisson process. **F**
2. The stochastic process $\{N(n) : n \geq 0\}$ is a (discrete-time) renewal counting process. **T**
3. The stochastic process $\{N(n) : n \geq 0\}$ is a Markov chain. **F**
4. The stochastic process $\{N(n) : n \geq 0\}$ is a periodic Markov chain. **F**

5. The stochastic process $\{N(n) : n \geq 0\}$ has independent increments. **F**
6. $E[N(n)]/n$ necessarily converges to a positive limit as $n \rightarrow \infty$. **T**, by the elementary renewal theorem

Scoring: 0 wrong: 5 points, 1 wrong: 3 points, 2 wrong: 0 points, ≥ 3 wrong: -1 point.

- (c) What is the probability that the king is again at its initial square after three moves?

The king can do anything in his first move. Given the first move he has to stay in the same row or column he moved in, but he cannot return to the starting square. That makes 6 of the 14 possible moves OK. Then in the third move only the target 1 move out of the possible 14 is OK. So the desired probability is

$$\frac{6}{14} \times \frac{1}{14} = \frac{3}{98}$$

- (d) What is the long-run proportion of moves that the king is at its initial square?

This is like problem 4.76 in the book, which was one of the sample-entertainment problems. We use the structure of a random walk on a graph. The reversibility implies that the steady-state probability is easy to compute. The long-run proportion is the number of moves possible from that initial square divided by the sum of the number of moves from each square, summed over all 64 squares. There are 4 corner squares where the king has 3 moves. There are $6 \times 4 = 24$ side squares where the king has 5 moves, and there are 36 interior squares where the king has 8 moves. Hence the long-run proportion is

$$\frac{3}{(4 \times 3) + (24 \times 5) + (36 \times 8)} = \frac{3}{420} = \frac{1}{140}.$$

- (e) What is the expected number of moves until the king first returns to its initial square?

The expected number of moves until returning is the reciprocal of the answer in part (d). The expected number is 140 moves.

- (f) Give an expression (formula or formulas) for the expected number of moves until the king first visits the opposite corner square?

We convert the Markov chain to an absorbing Markov chain by making the opposite corner square an absorbing state. We then use absorbing Markov chain theory. We represent the matrix in the form

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix},$$

where I is an identity matrix (1's on the diagonal and 0's elsewhere) and 0 (zero) is a matrix of zeros. In this case, I would be 1×1 , R is 63×1 and Q is 63×63). The matrix Q describes the probabilities of motion among the transient states, while the matrix R gives the probabilities of absorption in one step (going from one of the transient states to one of the absorbing states in a single step). We then proceed to compute the fundamental matrix

$$N = (I - Q)^{-1}$$

Assuming that our initial state is the first state in the numbering, then the expected number of steps before absorption (reaching the opposite corner) would be the sum of the elements in the first row in the matrix N so computed.

2. Back and Forth to Campus (20 points)

Professor Prhab Hubilliti lives at the bottom of the hill on the corner of 117th Street and 7th Avenue. Going each way - up hill to to teach his class at Columbia or down hill back home - Prhab either runs or walks. Going up the hill, Prhab either walks at 2 miles per hour or runs at 5 miles per hour. Going down the hill, Prhab either walks at 3 miles per hour or runs at 6 miles per hour. In each direction, he always runs the entire way or walks the entire way. Since Prhab often works late into the night, he often gets up late, and has to run up hill to get to his class. On any given day, Prhab runs up hill with probability $2/3$ and walks up hill with probability $1/3$. On the other hand, Prhab is less likely to run going back home. On any given day, he runs down hill with probability $1/3$ and walks down hill with probability $2/3$.

(a) What is the average speed Prhab goes up the hill to campus? What is the average speed Prhab goes down the hill back home?

The average speed each way is easy to compute: The average speed on each trip up hill is

$$(2/3)5 + (1/3)2 = \frac{12}{3} = 4,$$

while the average speed back down is

$$(2/3)3 + (1/3)6 = \frac{12}{3} = 4.$$

Hence, Prhab's average speed is 4 miles per hour in each direction.

(b) What is the long-run proportion of Prhab's total travel time going to and from campus that he spends going up hill to campus?

This is a variant of problem 7.38 in the book, which was yet another of the sample-entertainment problems. A key idea is to use "Dirt" (distance equals rate times time or $D = r \times T$). Here we are concerned about the times, not the rates. In particular, the time is the distance divided by the rate. We do not need to know the distance because it is the same each way. We can represent the total travel time as an alternating renewal process. He first goes up the hill to campus and then he returns down the hill back to his apartment. Let U be

the time to go up and let D be the time to go down. Then by renewal theory (renewal-reward processes), the long-run average of time going up hill to campus is

$$\text{proportion of time going up} = \frac{E[U]}{E[U] + E[D]},$$

where

$$E[U] = (2/3)\frac{d}{5} + (1/3)\frac{d}{2} = \frac{9d}{30} = \frac{3d}{10}$$

and

$$E[D] = (2/3)\frac{d}{3} + (1/3)\frac{d}{6} = \frac{5d}{18}.$$

Hence,

$$\text{proportion of time going up} = \frac{E[U]}{E[U] + E[D]} = \frac{(3/10)}{(3/10) + (5/18)} = \frac{(27/90)}{(27/90) + (25/90)} = \frac{27}{52}.$$

(c) What is the long-run proportion of Prhab's total travel time going to and from campus that he spends running up hill to campus?

We again use renewal reward processes. Now we want

$$\text{proportion of time running up} = \frac{E[U, \text{running}]}{E[U] + E[D]},$$

and use

$$E[U, \text{running}] = (2/3)\frac{d}{5}.$$

Hence, we get

$$\text{proportion of time running up} = \frac{E[U, \text{running}]}{E[U] + E[D]} = \frac{(12/90)}{(52/90)} = \frac{3}{13}.$$

3. The Boberg-Kapoor Investment Company (25 points)

Ross Boberg and Arjun Kapoor have been so successful with their Boberg-Kapoor Investment Company that their company has gone public and is now listed on the New York Stock Exchange under the BK symbol. Suppose that a share of BK stock initially cost \$100. Suppose that the BK share price over time (measured in years) is modelled as the stochastic process

$$S(t) \equiv 100 + 10B(t),$$

where $\{B(t) : t \geq 0\}$ is standard Brownian motion, having mean $E[B(t)] = 0$ and variance $Var(B(t)) = t$ for $t \geq 0$.

(a) What is the probability that the BK share price exceeds \$120 after 4 years?

Note that $S(t) \stackrel{d}{=} N(100, 100t)$, so $S(4) \stackrel{d}{=} N(100, 400)$. So

$$P(S(4) > 120) = P\left(\frac{S(4) - 100}{20} > \frac{120 - 100}{20}\right) = P(N(0, 1) > 1) = 0.16$$

from the table on page 81.

(b) What is the probability that the BK share price will exceed \$120 any time during the first 4 years?

We use the reflection principle, as on pages 629-630.

$$P\left(\max_{0 \leq t \leq 4} \{S(t)\} > 120\right) = 2P(S(4) > 120) \approx 2 \times 0.16 = 0.32$$

(c) What is the probability that the BK share price rises to \$140 before it drops to \$80, and then after hitting \$140, rises further to \$180 before it drops back to \$120?

This is a variant of homework exercise 10.5. We have two independent events. The desired probability is

$$(20/(20 + 40))(20/(20 + 40)) = (1/3)^2 = 1/9.$$

(d) What is the expected time until the BK share price first either rises to \$140 or drops to \$80?

Now we are using the formula for the expected time, $E[T]$, in the gambler's ruin problem for Brownian motion without drift; see the notes for Tuesday, December 4. The formula is derived by applying the optional stopping theorem (OST) to the quadratic martingale $\{B(t)^2 - t : t \geq 0\}$. The formula (for the time to go up x or down w) is

$$E[T] = \frac{xw}{\sigma^2} = \frac{20 \times 40}{100} = 8.$$

Translating to ordinary Brownian motion, we get the same answer:

$$E[T] = \frac{xw}{\sigma^2} = \frac{2 \times 4}{1} = 8.$$

(e) What is the conditional expectation:

$$E[S(2)S(3)|S(1) = 100]?$$

Using the stationary and independent increments property, we see that this question is equivalent to

$$E[S(1)S(2)|S(0) = 100]?$$

which in turn is equivalent to just

$$\begin{aligned} E[S(1)S(2)] &= E[S(1)^2] + E[S(1)(S(2) - S(1))] \\ &= E[S(1)^2] + E[S(1)]E[(S(1))] \\ &= \text{Var}(S(1)) + 2(E[S(1)])^2 = 100 + 2(100)^2 = 20,100. \end{aligned}$$

4. The DDT Barbershop (25 points)

Nishi Dedania, Rahul Dhir and Arita Thatte have joined together to form the DDT barbershop. The DDT barbershop has room for at most five customers, with up to three in service and two waiting. Suppose that potential customers arrive according to a Poisson process at a rate of 8 per hour. Suppose that potential arrivals finding the barber shop full, with three customers in service and two other customers waiting, will leave and not affect future arrivals. Suppose that successive service times are independent exponential random variables with mean 15 minutes. Suppose that waiting customers have limited patience, with each waiting customer being willing to wait only an independent random, exponentially distributed, time with mean 15 minutes before starting service; if the customer has not started service by that time, then the customer will abandon, leaving without receiving service.

(a) What proportion of time are all three barbers busy in the long run?

This is a continuous-time Markov chain (CTMC), specifically a birth-and-death (BD) process. The model is specified by the rate matrix Q , which for a BD process means the birth rates and death rates. Letting $\lambda = 8$ be the total arrival rate, the birth rates are

$$\lambda_k = \lambda = 8 \quad \text{for all } k, \quad 0 \leq k \leq 4,$$

while, letting $\mu = 4$ be the individual service rate and $\theta = 4$ be the individual abandonment rate, the death rates are

$$\mu_1 = \mu = 4, \mu_2 = 2\mu = 8, \mu_3 = 3\mu = 12, \mu_4 = 3\mu + \theta = 12 + 4 = 16, \mu_5 = 3\mu + 2\theta = 12 + 8 = 20.$$

By the standard formula, the steady state probabilities are

$$\alpha_j = \frac{r_j}{\sum_{k=0}^5 r_k}, \quad \text{for } 0 \leq j \leq 5,$$

where

$$\begin{aligned} r_0 &= 1, & r_1 &= \frac{\lambda_0}{\mu_1} = 2, & r_2 &= r_1 \frac{\lambda_1}{\mu_2} = 2 * 1 = 2, \\ r_3 &= r_2 \frac{\lambda_2}{\mu_3} = 2 * (2/3) = \frac{4}{3}, & r_4 &= r_3 \frac{\lambda_3}{\mu_4} = (4/3) * (1/2) = \frac{2}{3}, \\ r_5 &= r_4 \frac{\lambda_4}{\mu_5} = (2/3) * (8/20) = \frac{4}{15}, \end{aligned}$$

Hence,

$$\begin{aligned} \alpha_0 &= \frac{15}{109}, & \alpha_1 &= \frac{30}{109}, & \alpha_2 &= \frac{30}{109}, & \alpha_3 &= \frac{20}{109} \\ \alpha_4 &= \frac{10}{109}, & \alpha_5 &= \frac{4}{109}. \end{aligned}$$

Thus, the proportion of time are all three barbers busy in the long run is

$$\alpha_3 + \alpha_4 + \alpha_5 = \frac{34}{109}$$

(b) What is the total rate of customer abandonment in the long run?

$$\begin{aligned}\text{rate} &= \alpha_4(\theta) + \alpha_5(2\theta) \\ &= \frac{10}{109} \times 4 + \frac{4}{109} \times 8 \\ &= \frac{40}{109} + \frac{32}{109} = \frac{72}{109}\end{aligned}$$

(c) What proportion of all potential arrivals are served?

The rate customers are served is

$$\begin{aligned}\text{rate} &= \alpha_1\mu + \alpha_2(2\mu) + (\alpha_3 + \alpha_4 + \alpha_5)(3\mu) \\ &= \frac{30}{109} \times 4 + \frac{30}{109} \times 8 + \frac{34}{109} \times 12 \\ &= \frac{768}{109}.\end{aligned}$$

The rate of potential arrivals is $\lambda = 8$ per hour.

Hence the proportion of all potential arrivals that are served is the ratio of these two rates:

$$\text{proportion} = \frac{(768/109)}{8} = \frac{96}{109}.$$

(d) What is the probability that, starting empty, the third customer arrives before anybody has completed service?

Let U_1 be the arrival time of the first customer. Let U_k be the interarrival time between the $(k-1)^{\text{st}}$ customer and the k^{th} customer. Let V_k be the k^{th} service time. Then the desired probability is

$$P(U_2 < V_1)P(U_3 < \min\{V_1^*, V_2\}),$$

where all these random variables are independent exponential random variables. In particular, by the lack of memory property, V_1^* is a random variable distributed the same as V_1 , but independent of it and all the other random variables. But recall that U_k is exponential with rate 8, while V_k is exponential with rate 4. Hence, the desired probability is

$$\frac{8}{8+4} \times \frac{8}{8+8} = (2/3) \times (1/2) = \frac{1}{3}.$$

(e) What is the probability that, starting empty, the fourth customer arrives and abandons before anybody has completed service?

We have to have the above in part (d) plus more. In particular, the probability is:

$$P(U_2 < V_1)P(U_3 < \min \{V_1^*, V_2\})P(U_4 < \min \{V_1^{**}, V_2^*, V_3\})P(T_4 < \min \{V_1^{***}, V_2^{**}, V_3^*\}),$$

where T_4 is the exponential time to abandon for customer 4 if he is waiting and the asterisks are used to indicate new independent random variables with the same distribution. Hence the desired probability is

$$\frac{1}{3} \times \frac{8}{8+12} \times \frac{4}{4+12} = \frac{1}{3} \times \frac{2}{5} \times \frac{1}{4} = \frac{1}{30}.$$
