IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2012, Professor Whitt

SOLUTIONS for Homework Assignment 11

Chapter 10: Brownian Motion

In Ross, read Sections 10.1-10.3 and 10.6. (The total required reading is approximately 11 pages.)

Do the following exercises at the end of Chapter 10.

1. (answer in back)

This seems simple, but it is not quite as simple as it appears. First, $B(s) \stackrel{d}{=} N(0,s)$, i.e., $B(s)$ is normally distributed with mean 0 and variance s. Similarly, $B(t) \stackrel{d}{=} N(0,t)$. The difficulty is that these two random variables $B(s)$ and $B(t)$ are DEPENDENT. So we cannot compute the distribution of the sum by doing a convolution.

However, we can exploit the independent increments property to rewrite the sum as a sum of independent random variables. By adding and subtracting $B(s)$, we have $B(t) = B(s) + [B(t) - B(s)]$, where $B(t) - B(s)$ is independent of $B(s)$ by the independent increments property of Brownian motion. Hence

$$B(s) + B(t) = 2B(s) + [B(t) - B(s)] \ . \tag{1}$$

This representation is better because it is the sum of two independent random variables. We could compute its distribution by doing a convolution, but we will use another argument.

In whatever way we proceed, we will want to use the stationary increments property to deduce that

$$B(t) - B(s) \stackrel{d}{=} B(t-s) - B(0) = B(t-s) \stackrel{d}{=} N(0, t-s) \ .$$

We now invoke a general property about multivariate normal distributions. A linear function of normal random variables is again a normal random variable. (This is true even with dependence. This is true in all dimensions.) We thus know that $B(s) + B(t)$ is normal or, equivalently, $2B(s) + [B(t) - B(s)]$ is normal.

However, we do not need that general result, because we can represent (1) above, which tells us that we have the sum of two independent normal random variables. We know that is normally distributed by virtue of Example 2.46 on p. 70 (see addendum at the end.). Either way, we know we have a normal distribution. A normal distribution is determined by its mean and variance. Since $E[B(t)] = 0$, it is elementary that

$$E[B(s) + B(t)] = 0 \quad \text{or} \quad E[2B(s) + [B(t) - B(s)]] = 0 \ .$$

Hence, finally, it suffices to compute the variance of $B(s)+B(t) = 2B(s)+[B(t)-B(s)]$. The second representation is easier, because the random variables are independent. For independent random variables, the variance of a sum is the sum of the variances. Hence

$$Var(2B(s)+B(t-s)) = Var(2B(s))+Var(B(t-s)) = 4Var(B(s))+Var(B(t-s)) = 4s+(t-s) = 3s+t \ .$$

Hence $B(s)+B(t) \stackrel{\mathrm{d}}{=} N(0,3s+t)$, i.e., is normally distributed with mean 0 and variance $3s+t$.

ADDENDUM: Here is what Example 2.46 shows: We can directly deduce that the sum of two independent mean normal random variables is normally distributed. Suppose $X_i \stackrel{\mathrm{d}}{=} N(m_i, \sigma_i^2)$ for $i = 1, 2$. Then $X_i \stackrel{\mathrm{d}}{=} m_i + \sigma_i N_i(0,1)$. Thus, $X_1 + X_2 \stackrel{\mathrm{d}}{=} (m_1+m_2) + \sigma_1 N_1(0,1) + \sigma_2 N_2(0,1)$, where $N_1(0,1)$ and $N_2(0,1)$ are independent standard normal random variables. Using moment generating functions (mgf's), it is easy to see that

$$\phi_{N_1(0,1)+N_2(0,1)}(t) \equiv Ee^{t[N_1(0,1)+N_2(0,1)]} = E[e^{t[N_1(0,1)}]E[e^{tN_2(0,1)}] = [e^{t^2/2}]^2 = e^{t^2} \ ,$$

which is the mgf of a normal distribution.

2. The conditional distribution $X(s) - A$ given that $X(t_1) = A$ and $X(t_2) = B$ is the same as the conditional distribution of $X(s-t_1)$ given that $X(0) = 0$ and $X(t_2 - t_1) = B - A$, which (by eq. 10.4) is normal with mean $\frac{s-t_1}{t_2-t_1}(B-A)$ and variance $\frac{s-t_1}{t_2-t_1}(t_2 - s)$. Hence the desired conditional distribution is normal with mean $A + \frac{s-t_1}{t_2-t_1}(B-A)$ and variance $\frac{s-t_1}{t_2-t_1}(t_2 - s)$

3. (answer in back)

One approach is to use martingales: First, we can write the expectation as the expectation of a conditional expectation:

$$E[B(t_1)B(t_2)B(t_3)] = E[E[B(t_1)B(t_2)B(t_3)|B(r), 0 \le r \le t_2]] \ .$$

Now we evaluate the inner conditional expectation:

$$E[B(t_1)B(t_2)B(t_3)|B(r), 0 \le r \le t_2] = B(t_1)B(t_2)E[B(t_3)|B(t_2)] = B(t_1)B(t_2)^2 \ .$$

Hence the answer, so far, is

$$E[B(t_1)B(t_2)B(t_3)] = E[B(t_1)B(t_2)^2] \ .$$

We now proceed just as above by writing this expectation as the expectation of a conditional expectation:

$$E[B(t_1)B(t_2)B(t_3)] = E[B(t_1)B(t_2)^2] = E[E[B(t_1)B(t_2)^2|B(r), 0 \le r \le t_1]] \ .$$

We next evaluate the inner conditional expectation:

$$E[B(t_1)B(t_2)^2|B(r), 0 \le r \le t_1] = B(t_1)E[B(t_2)^2|B(t_1)] \ ,$$

where,

$$B(t_2)^2 = [B(t_1) + B(t_2) - B(t_1)]^2 = B(t_1)^2 + 2B(t_1)[B(t_2) - B(t_1)] + [B(t_2) - B(t_1)]^2 \ ,$$

so that

$$\begin{aligned} B(t_1)E[B(t_2)^2|B(t_1)] &= B(t_1)^3 + 2B(t_1)^2 E[B(t_2) - B(t_1)|B(t_1] + B(t_1)E[B(t_2) - B(t_1)]^2 \\ &= B(t_1)^3 + 2B(t_1)^2 \times 0 + B(t_1)E[B(t_2 - t_1)]^2 \\ &= B(t_1)^3 + B(t_1)(t_2 - t_1) \end{aligned}$$

Now taking expected values again, we get

$$\begin{aligned} E[B(t_1)B(t_2)B(t_3)] &= E[B(t_1)B(t_2)^2] = E[E[B(t_1)B(t_2)^2|B(r), 0 \le r \le t_1]] \\ &= E[B(t_1)^3] + E[B(t_1)(t_2 - t_1)] = 0 + 0 \ , \end{aligned}$$

using the fact that $E[B(t_1)^3] = 0$, because the third moment of a normal random variable with mean 0 is necessarily 0, because of the symmetry. Hence, $E[B(t_1)B(t_2)B(t_3)] = 0$.

Another longer, but less complicated (because it does not use conditional expectations), argument is to break up the expression into independent pieces: Replace $B(t_3)$ by $B(t_2) + [B(t_3) - B(t_2)]$ and then by $B(t_1) + [B(t_2) - B(t_1)] + [B(t_3) - B(t_2)]$. And replace $B(t_2)$ by $B(t_1) + [B(t_2) - B(t_1)]$. Then substitute into the original product and evaluate the expectation:

$$E[B(t_1)B(t_2)B(t_3)] = E[B(t_1)(B(t_1)+[B(t_2)-B(t_1)])(B(t_1)+[B(t_2)-B(t_1)]+[B(t_3)-B(t_2)])] .$$

Or, equivalently,

$$\begin{aligned} E[B(t_1)B(t_2)B(t_3)] &= E[x_1(x_1 + x_2)(x_1 + x_2 + x_3)] \\ &= E[x_1^3 + 2x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3] , \end{aligned}$$

where $x_1 \equiv B(t_1)$, $x_2 = B(t_2) - B(t_1)$ and $x_3 = B(t_3) - B(t_2)$. Since the random variables x_1, x_2 and x_3 are independent (the point of the construction),

$$E[x_1^3 + 2x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3] = E[x_1^3] = E[B(t_1)^3] = 0 ,$$

because the third moment of a normal random variable with mean 0 is 0.

5. (answer in back)

Let $T_x \equiv \min\{t \geq 0 : B(t) = x\}$. What is

$$P(T_1 < T_{-1} < T_2)?$$

The event $\{T_1 < T_{-1}\}$ says that the Brownian motion B hits state 1 before hitting -1. The probabilities of hitting each of the boundaries can be obtained by using martingales, as shown in the class notes. Given that that event occurs (with the Brownian motion starting at state 1), the event $\{T_{-1} < T_2\}$ is equivalent to the event $\{T_{-2} < T_1\}$ for the original Brownian motion (starting at 0 instead of at 1). Moreover, these events are independent. Hence, we can write

$$P(T_1 < T_{-1} < T_2) = P(T_1 < T_{-1})P(T_{-2} < T_1) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} .$$

6. The probability of recovering your purchase price is the probability that a Brownian Motion goes up c by time t. Hence the desired probability is

$$1 - P\{\max_{0 \leq s \leq t} X(s) \geq c\} = 1 - \frac{2}{\sqrt{2\pi}} \int_{\frac{a}{\sqrt{t}}}^{\infty} e^{-y^2/2} dy$$

7. We can find this probability by conditioning on $X(t_1)$

$$P\{\max_{t_1 \leq s \leq t_2} X(s) > x\} = \int_{-\infty}^{\infty} P\{\max_{t_1 \leq s \leq t_2} X(s) > x | X(t_1) = y\} \frac{1}{\sqrt{2\pi t_1}} e^{y^2/2t_1} dy \ (*)$$

Where

$$\begin{aligned} P\{\max_{t_1 \leq s \leq t_2} X(s) > x | X(t_1) = y\} &= P\{\max_{0 \leq s \leq t_2 - t_1} X(s) > x - y\} \text{ if } y < x \\ &= 1 \text{ if } y > x \end{aligned}$$

Substitution of the above equation into (*) now gives the required result when one uses the following,

$$P\{\max_{0 \le s \le t_2 - t_1} X(s) > x - y\} = 2P\{X(t_2 - t_1) > x - y\}$$

Where $X(t_2 - t_1) \sim N(0, t_2 - t_1)$

16. Taking expectations of the defining equation of a Martingale yields

$$E[Y(t)] = E[E[Y(t)|Y(u), 0 \le u \le s]] = E[Y(s)]$$

That is, $E[Y(t)]$ is a constant and so is equal to $E[Y(0)]$

17. To prove that standard Brownian Motion is a Martingale we just prove that it satisfies the condition for Martingales given in the above question i.e. $E[Y(t)|Y(u), 0 \le u \le s] = Y(s)$

$$E[B(t)|B(u), 0 \le u \le s] = E[B(s) + B(t) - B(s)|B(u), 0 \le u \le s]$$

$$= E[B(s)|B(u), 0 \le u \le s] + E[B(t) - B(s)|B(u), 0 \le u \le s] = B(s) + E[B(t) - B(s)] = B(s)$$

Where the second to last equality follows from the independent increment property of Brownian Motion.

20. (answer in back)

21. By the Martingale stopping theorem $E[B(T)] = E[B(0)] = 0$
But $B(T) = \frac{x - \mu T}{\sigma}$ and so $E[\frac{x - \mu T}{\sigma}] = 0$ and hence $E[T] = x/\mu$.