

IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2012, Professor Whitt

Solutions to Homework Assignment 5.

Do the following exercises at the end of Chapter 5:

1. Since T is exponential with mean $1/2$ hour,

$$P(T > t) = e^{-2t}, \quad t > 0 .$$

Hence,

(a)

$$P(T > 1/2) = e^{-2(1/2)} = e^{-1} \approx \frac{1}{2.71828} .$$

(b) By the lack-of-memory property (see Section 5.2.2),

$$P(T > 12.5 | T > 12) = P(T > 1/2) = e^{-2(1/2)} = e^{-1} .$$

2. The time you spend in the system, say T , is the sum of 6 IID (independent and identically distributed) exponential random variables, each with mean $1/\mu$. Hence,

$$ET = 6/\mu .$$

3. The conditional distribution of X given that $X > 1$ is the same as the unconditional distribution of $1 + X$. Hence

$$E[X^2 | X > 1] = E[(X + 1)^2] ;$$

i.e., (a) is correct.

4. (a) 0, (b) one way: $(1/3)^3 = 1/27$, (c) $1/4$ because

$$\begin{aligned} (A \text{ last to leave}) &= P(B \text{ served before } A) \times P(C \text{ served before } A | B \text{ served before } A) \\ &= (1/2) \times (1/2) = 1/4 . \end{aligned}$$

5. Let T be the lifetime of the radio. Then

$$P(T > t \text{ years}) = e^{-0.1t}, \quad t > 0 .$$

Hence, by the lack-of-memory property (again see Section 5.2.2),

$$P(T > 20|T > 10) = P(T > 10) = e^{-0.1(10)} = e^{-1} .$$

8. Note that

$$\begin{aligned} E[1/r(X)] &= \int (1/r(x))f(x) dx \\ &= \int \frac{1 - F(x)}{f(x)} f(x) dx \\ &= \int 1 - F(x) dx \\ &= E[X] . \end{aligned}$$

The last step is a very useful relation, as indicated in the hint; see page 580 of Ross.

9. The probability that machine 1 fails before time t is $1 - e^{-\lambda_1 t}$. The probability that machine 1 is still working at time t is $e^{-\lambda_1 t}$. Conditional on machine 1 working at time t , the probability that machine 1 fails first is $\lambda_1/(\lambda_1 + \lambda_2)$. Hence the final answer is

$$1 - e^{-\lambda_1 t} + e^{-\lambda_1 t} \frac{\lambda_1}{(\lambda_1 + \lambda_2)} .$$

11. Use two facts: (1) the lack of memory property and (2) the fact that the minimum of independent exponential random variables is again exponential with a rate equal to the sum of the individual rates (rate equals reciprocal of the mean). Hence,

$$P(A_1) = P(X > \min\{Y_1, \dots, Y_n\}) = \frac{n\mu}{\lambda + n\mu} .$$

Moreover,

$$P(A_j|A_1 A_2 \dots A_{j-1}) = \frac{(n - j + 1)\mu}{\lambda + (n - j + 1)\mu} .$$

Hence,

$$\begin{aligned} p &\equiv P(X > \max\{Y_i\}) \\ &= P(A_1)P(A_2|A_1) \dots P(A_n|A_1 A_2 \dots A_{n-1}) \\ &= \prod_{j=1}^{j=n} \frac{(n - j + 1)\mu}{\lambda + (n - j + 1)\mu} . \end{aligned}$$

When $n = 2$,

$$\begin{aligned} p &\equiv P(X > \max\{Y_i\}) \\ &= \int_0^{\infty} P(\max\{Y_i\} < X|X = x)\lambda e^{-\lambda x} dx \\ &= \int_0^{\infty} P(\max\{Y_i\} < x)\lambda e^{-\lambda x} dx \\ &= \int_0^{\infty} (1 - e^{-\mu x})^2 \lambda e^{-\lambda x} dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty (1 - 2e^{-\mu x} + e^{-2\mu x})\lambda e^{-\lambda x} dx \\
&= 1 - \frac{2\lambda}{\lambda + \mu} + \frac{\lambda}{\lambda + 2\mu} \\
&= \frac{2\mu^2}{(\lambda + \mu)(\lambda + 2\mu)}.
\end{aligned}$$

26. This is an exercise in properties 2, 4 and 5 on the Concise Summary: (2) The minimum of independent exponential random variables is again exponential with a rate equal to the sum of the rates of the random variables considered in the minimum; (4) the probability that one exponential random variable is the minimum of several independent exponential random variables is its rate divided by the sum of the rates; (5) even more, the two events considered in (2) and (4) are actually independent! However, it is necessary to assume that all these exponential random variables are mutually independent. We make that assumption.

(a) This is property 4 above. This is just the probability one exponential variable is less than another:

$$\frac{\mu_1}{\mu_1 + \mu_3}$$

(b) This is property 4 applied twice. You now have two independent events that must both happen, so it is the product of the probabilities:

$$\frac{\mu_1}{\mu_1 + \mu_3} \times \frac{\mu_2}{\mu_2 + \mu_3}$$

(c) Now you want to be very careful. You want to exploit all three properties. You will need the more complicated property (5). Let T be the sojourn time of the new customer. To compute $E[T]$, we condition on the first event, i.e., whether the new customer finishes service at server 1 first ($S_1 < S_3$) or the old customer finishes service at service 3 first ($S_1 > S_3$). Let T_0 be the time up to the first event. Since $T_0 = \min(S_1, S_3)$, $E[T_0] = \frac{1}{\mu_1 + \mu_3}$. Let $T' = T - T_0$, the remaining sojourn time of the new customer. So,

$$\begin{aligned}
E[T] &= E[T_0] + E[T'|S_1 > S_3]P(S_1 > S_3) + E[T'|S_1 < S_3]P(S_1 < S_3) \\
&= \left(\frac{1}{\mu_1 + \mu_3}\right) + \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}\right)\left(\frac{\mu_3}{\mu_1 + \mu_3}\right) + E[T'|S_1 < S_3]\left(\frac{\mu_1}{\mu_1 + \mu_3}\right)
\end{aligned}$$

To review a key step, the event $S_1 > S_3$ is independent of T_0 . Hence above we could just use $P(S_1 > S_3)$ in the second term.

To obtain $E[T'|S_1 < S_3]$, we first notice that this is the scenario that the new customer starts service at server 2 and the old customer remains in service at server 3 since the new customer enters service at server 2 immediately right after he finishes service at server 1, and we will condition on whether the new customer finishes service at server 2 first ($S_2 < S_3$) or the old customer finishes service at server 3 first ($S_2 > S_3$). By the lack of memory property, the remaining service time of the old customer at server 3 is still $\exp(\mu_3)$. Let T'_0 be the time up to this first event, i.e., $T'_0 = \min(S_2, S_3)$. Then $E[T'_0] = \frac{1}{\mu_2 + \mu_3}$. Let $T'_r = T' - T'_0$, the remaining service sojourn time. So,

$$\begin{aligned}
E[T'|S_1 < S_3] &= E[T'_0] + E[T'_r|S_2 > S_3]P(S_2 > S_3) + E[T'_r|S_2 < S_3]P(S_2 < S_3) \\
&= \left(\frac{1}{\mu_2 + \mu_3}\right) + \left(\frac{1}{\mu_2} + \frac{1}{\mu_3}\right)\left(\frac{\mu_3}{\mu_2 + \mu_3}\right) + \frac{2}{\mu_3}\left(\frac{\mu_2}{\mu_2 + \mu_3}\right).
\end{aligned}$$

(d) Let X_i be the service time at server i for customer X (new customer), $i = 1, 2, 3$. Let Y_j be the service time at server j for customer Y (old customer), $j = 2, 3$. We know that X_i 's and Y_j 's are independent and exponentially distributed (but with different means). Let T be the total time required. Let T_1 = time until next event happens, and let $T' = T - T_1$. Then $ET = ET_1 + ET'$. Clearly $T_1 = \min(X_1, Y_2)$, so $ET_1 = \frac{1}{\mu_1 + \mu_2}$. We remark that the distribution of T_1 is independent of the event $\{X_1 < Y_2\}$; see property 5 of concise summary.) Now we condition to find the mean of T' :

$$\begin{aligned} ET' &= E[T'|X_1 > Y_2] \cdot P(X_1 > Y_2) + E[T'|X_1 < Y_2] \cdot P(X_1 < Y_2) \\ &= E[T'|X_1 > Y_2] \cdot \left(\frac{\mu_2}{\mu_1 + \mu_2}\right) + E[T'|X_1 < Y_2] \cdot \left(\frac{\mu_1}{\mu_1 + \mu_2}\right) \end{aligned}$$

Note that $E[T'|X_1 > Y_2]$ is already computed in part (c), the only unknown here is $E[T'|X_1 < Y_2]$. Note that

$$E[T'|X_1 < Y_2] = \frac{1}{\mu_2} + ET_2 + ET''$$

Where T_2 = the time until next event happens after customer Y finishes his service at server 2, then T'' = the time from the next event happens until customer X leaves the system. Since $T_2 = \min(X_2, Y_3)$, we have $ET_2 = \frac{1}{\mu_2 + \mu_3}$. And also:

$$\begin{aligned} E[T''] &= E[T''|X_2 > Y_3] \cdot P(X_2 > Y_3) + E[T''|X_2 < Y_3] \cdot P(X_2 < Y_3) \\ &= \left(\frac{1}{\mu_2} + \frac{1}{\mu_3}\right) \cdot \left(\frac{\mu_3}{\mu_2 + \mu_3}\right) + \frac{2}{\mu_3} \cdot \left(\frac{\mu_2}{\mu_2 + \mu_3}\right) \end{aligned}$$

It's not necessary to simplify this result.