IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2012, Professor Whitt

Homework Assignment 6: Tuesday, October 16

Due on Tuesday, October 23.

Chapter 6: Continuous-Time Markov Chains

Read Sections 6.1-6.5 in Ross. Do the following exercises at the end of Chapter 6.

You are asked to turn in the problems without answers in the back, but you should do all the problems.

2. (Answer in back)

Remarks: The object here is to construct a continuous-time Markov chain (CTMC) model. There are two key steps: (1) defining the state space and (2) specifying the local transition structure. When we define the state space, we can define the CTMC stochastic process taking values in that state space. The state space is discrete, and so can be identified with a subset of the integers, but here it is better to use pairs of integers. The question in applications is: What are the states?

The "local" transition structure of a CTMC can be defined in two ways. First, following Section 6.2 of Ross, we can specify the transition matrix P for the embedded discrete-time Markov chain (DTMC) operating at the at the transition epochs of the CTMC and the rates of the exponential holding times in each state. As in Ross, let ν_i denote the rate of the exponential holding time in state i; i.e., the holding time in state i has an exponential distribution with mean $1/\nu_i$. Thus, it suffices to specify the matrix P and the vector ν.

An alternative way to specify a CTMC is to specify the (infinitesimal) transition rates. The (infinitesimal) transition rate of a transition from state i to state j is denoted by $Q_{i,j}$. (Ross uses the lower-case $q_{i,j}$; we will use both.) The diagonal entries $Q_{i,i}$ of the matrix Q can be unspecified or can be defined as $Q_{i,i} \equiv -\sum_{j,j\neq} Q_{i,j} = -\nu_i$. In terms of the model elements P and ν introduced above, we have

$$Q_{i,j} = \nu_i P_{i,j} \quad \text{for} \quad j \neq i \ .$$

We can define Q from (P, ν), and we can define (P, ν) given Q. Thus we have two alternative ways to specify a CTMC.

3.

Remarks: Recall that a birth-and-death (BD) process is a CTMC defined on a subset of the nonnegative integers that moves only to neighboring states; i.e., from state i it can only go next to one of $i-1$ or $i+1$. The transition rate from i to $i+1$ is denoted by λ_i instead of $Q_{i,i+1}$, as it would be in the CTMC representation. Similarly, the transition rate from i to $i-1$ is denoted by μ_i instead of by $Q_{i,i+1}$, as it would be in the CTMC representation. In other words, we can use the two ways to specify a CTMC indicated above in the remark about Problem 2 above, but we can also use the special way for BD processes involving the birth rates λ_i and the death rates μ_i.

Our goal here is to construct the Markov chain model, not solve it.

9.

13. Hint: See Section 6.5.

20.

23.