

# IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2012, Professor Whitt

## Solutions to Homework Assignment 7: Due on Tuesday, October 30.

More of Chapter 5: Read the rest of Section 5.3, skipping Examples 5.17, 5.21 and Subsection 5.3.6. Read Section 5.4 up to (but not including) Example 5.27. Poisson process and nonhomogeneous Poisson process.

31. Let  $T$  be the amount of time (minutes) that the 1:30 pm appointment spends at the doctor's office. Let the event  $A \equiv \{1 \text{ pm appointment leaves before 1:30pm}\}$  and the event  $A^c \equiv \{1 \text{ pm appointment leaves after 1:30pm}\}$ . Denote the service times of the 1pm appointment and 1:30pm appointment by  $S_1$  and  $S_2$ , respectively. Then by assumption  $S_1$  and  $S_2$  are iid  $\exp(\frac{1}{30})$ . By conditioning on whether the 1 pm appointment has left or not when the 1:30 pm appointment arrives on time, we obtain

$$\begin{aligned} E[T] &= E[T|A]P(A) + E[T|A^c]P(A^c) \\ &= 30P(S_1 \leq 30) + (30 + 30)P(S_1 > 30) \\ &= 30(1 + P(S_1 > 30)) \\ &= 30(1 + e^{-\frac{1}{30}30}) = 30(1 + e^{-1}). \end{aligned}$$

34. Let  $T_A$  and  $T_B$  be the life time of the two individuals  $A$  and  $B$ , respectively. Let  $T_1$  be the time until the first arrival of kidney and  $T_2$  be the time between the first arrival and the second arrival of kidneys. Then  $T_A \sim \exp(\mu_A)$  and  $T_B \sim \exp(\mu_B)$ ,  $T_i \sim \exp(\lambda)$ ,  $i = 1, 2$ . All these variables are mutually independent.

(a) The event that A obtains a new kidney happens when the first kidney arrives before A dies, i.e.  $T_1 < T_A$ , so

$$P(\text{A obtains a new kidney}) = P(T_1 < T_A) = \frac{\lambda}{\lambda + \mu_A}.$$

(b) To obtain the probability of B obtains a new kidney, we will condition on the first event, i.e., which happens first: a kidney arrives, A dies or B dies. Thus

$$\begin{aligned} P(\text{B obtains a new kidney}) &= P(\text{B obtains a new kidney} | T_1 = \min\{T_1, T_A, T_B\})P(T_1 = \min\{T_1, T_A, T_B\}) \\ &\quad + P(\text{B obtains a new kidney} | T_A = \min\{T_1, T_A, T_B\})P(T_A = \min\{T_1, T_A, T_B\}) \\ &\quad + P(\text{B obtains a new kidney} | T_B = \min\{T_1, T_A, T_B\})P(T_B = \min\{T_1, T_A, T_B\}) \\ &= P(T_2 < T_B)P(T_1 = \min\{T_1, T_A, T_B\}) + P(T_1 < T_B)P(T_A = \min\{T_1, T_A, T_B\}) + 0 \\ &= \left(\frac{\lambda}{\lambda + \mu_B}\right)\left(\frac{\lambda}{\lambda + \mu_A + \mu_B}\right) + \left(\frac{\lambda}{\lambda + \mu_B}\right)\left(\frac{\mu_A}{\lambda + \mu_A + \mu_B}\right) \\ &= \left(\frac{\lambda}{\lambda + \mu_B}\right)\left(\frac{\lambda + \mu_A}{\lambda + \mu_A + \mu_B}\right). \end{aligned}$$

39. Recall Problem 4 (e) on the midterm exam. We can use a normal approximation by virtue of the central limit theorem. See Section 2.7 if you need a refresher. By the assumptions,

the lifetime is the sum of 196 IID exponential random variables each with mean 1/2.5 years, i.e.,

$$S_{196} = X_1 + \cdots + X_{196} ,$$

where  $EX_i = 1/2.5$ . Thus,

(a) the mean is

$$E[S_{196}] = 196/2.5 = 78.4 .$$

(b) and the variance is

$$Var[S_{196}] = 196(1/2.5)^2 = 31.36 .$$

For parts (c) - (e), use the normal approximation, which follows from the central limit theorem; see Section 2.7. Use the normal probabilities in Table 2.3. Let  $Z$  be a standard normal random variable (with mean 0 and variance 1).

(c)

$$P(S_{196} < 67.2) \approx P(Z < (67.2 - 78.4)/5.6) = P(Z < -2.0) = 0.0227 .$$

(d)

$$P(S_{196} > 90) \approx P(Z > (90 - 78.4)/5.6) = P(Z > 2.07) = 0.0192 .$$

(e)

$$P(S_{196} > 100) \approx P(Z > (100 - 78.4)/5.6) = P(Z > 3.857) = 0.00006 .$$

42.

(a) This is just the sum of four IID exponential random variables:

$$E[S_4] = \frac{4}{\lambda} .$$

(b) The conditioning event says that there are two events up to time 1. Hence,

$$E[S_4|N(1) = 2] = 1 + E[\text{time for } 2 \text{ more events}] = 1 + \frac{2}{\lambda} .$$

(c) Recall that a Poisson process has independent increments. So,

$$E[N(4) - N(2)|N(1) = 3] = E[N(4) - N(2)] = E[N(2) - N(0)] = E[N(2)] = 2\lambda .$$

60. Let  $\{N(t), t \geq 0\}$  be the Poisson process with rate  $\lambda$ . Then by assumption  $N(1) = 2$ .

(a) The probability that both arrived during the first 20 minutes is

$$\begin{aligned} & P(N(1/3) = 2|N(1) = 2) \\ = & \frac{P(N(1/3) = 2, N(1) = 2)}{P(N(1) = 2)} \end{aligned}$$

$$\begin{aligned}
&= \frac{P(N(1/3) = 2, N(1) - N(1/3) = 0)}{P(N(1) = 2)} \\
&= \frac{P(N(1/3) = 2)P(N(1) - N(1/3) = 0)}{P(N(1) = 2)} \quad \text{by independent increments} \\
&= \frac{e^{-\frac{1}{3}\lambda} \frac{(\frac{1}{3}\lambda)^2}{2} e^{-\frac{2}{3}\lambda}}{e^{-\lambda} \frac{\lambda^2}{2}} \\
&= \frac{1}{9}.
\end{aligned}$$

(b) The probability that at least one arrived during the first 20 minutes is

$$\begin{aligned}
&P(N(1/3) \geq 1 | N(1) = 2) \\
&= 1 - P(N(1/3) = 0 | N(1) = 2) \\
&= 1 - \frac{P(N(1/3) = 0, N(1) = 2)}{P(N(1) = 2)} \\
&= 1 - \frac{P(N(1/3) = 0, N(1) - N(1/3) = 2)}{P(N(1) = 2)} \\
&= 1 - \frac{P(N(1/3) = 0)P(N(1) - N(1/3) = 2)}{P(N(1) = 2)} \quad \text{by independent increments} \\
&= 1 - \frac{e^{-\frac{1}{3}\lambda} e^{-\frac{2}{3}\lambda} \frac{(\frac{2}{3}\lambda)^2}{2}}{e^{-\lambda} \frac{\lambda^2}{2}} \\
&= 1 - \frac{4}{9} = \frac{5}{9}.
\end{aligned}$$

78. The number of customers that enter the store on a given day is Poisson with mean

$$\begin{aligned}
m(17) - m(8) &= \int_8^{17} \lambda(t) dt \\
&= \int_8^{10} 4 dt + \int_{10}^{12} 8 dt + \int_{12}^{14} (t - 4) dt + \int_{14}^{17} (-2t + 38) dt \\
&= 63
\end{aligned}$$

94. Let  $x$  be a point on the plane and  $B(x, r)$  be the circular region with radius  $r$  centered at  $x$ .  $X$  is the distance from  $x$  to its nearest event.

(a)

$$\begin{aligned}
P(X > t) &= P(\text{no event in } B(x, t)) \\
&= e^{-\lambda \text{Area}(B(x, t))} = e^{-\lambda \pi t^2}.
\end{aligned}$$

(b)

$$\begin{aligned}
E[X] &= \int_0^{\infty} P(X > t) dt \\
&= \int_0^{\infty} e^{-\lambda \pi t^2} dt \\
&= \sqrt{\frac{1}{\lambda}} \int_0^{\infty} \frac{1}{\sqrt{2\pi} \frac{1}{2\lambda\pi}} e^{-\frac{t^2}{2\lambda\pi}} dt
\end{aligned}$$

$$= \sqrt{\frac{1}{\lambda}} \times \frac{1}{2} = \frac{1}{2\sqrt{\lambda}}.$$