IEOR 3106: Professor Whitt

Lecture Notes, Thursday, September 20, 2012

More on Markov Chains

1. Liberating Markov Mouse: An Absorbing Markov Chain

It is important to see that there are **two basic kinds of markov chains**: (i) **irreducible Markov chains** and (ii) **reducible Markov chains**. In an irreducible Markov chain, it is possible to get to any other state in some finite number of moves; in a reducible Markov chain, it is not. Prominent among the reducible Markov chains are the **absorbing Markov chains**. In absorbing Markov chains there are states that the chain can enter, but once entered, the Markov chain cannot leave these states.

An example of an absorbing Markov chain appeared in the last part of the Markov mouse notes from the last class. We discussed that part of the previous notes.

The analysis is different for irreducible Markov chains and absorbing Markov chains. For irreducible Markov chains (but not for absorbing Markov chains), we have the fundamental equation $\pi = \pi P$. In contrast, for absorbing Markov chains, we have the fundamental matrix $N = (I - Q)^{-1}$ and the associated matrix of absorption probabilities $B = NR$.

It is important to recognize that these formulas and equations are being expressed concisely in **matrix notation**. For example, the equation $\pi = \pi P$ is a matrix equation; the π appearing there is a probability vector, while P is the square matrix of transition probabilities. If there are n states, then $\pi \equiv (\pi_1, \ldots, \pi_n)$ is a $1 \times n$ matrix (and thus a row vector), while P is an $n \times n$ square matrix with entries $P_{i,j}$. (Thus, the matrix multiplication πP is indeed well defined, and the product must itself be a $1 \times n$ row vector. Thus, the equation $\pi = \pi P$ corresponds to a system of equations that must be satisfied by the steady-state probability mass function, denoted by the probability vector π. Similarly, $N = (I - Q)^{-1}$ is a matrix inverse. (See the notes for the last class.)

2. A Bond Rating Example

I present this as a quite realistic model of a Markov chain from finance, which turns out to be an absorbing Markov chain. See the associated handout also posted on line.

3. Examples 4.1 and 4.4: Markov Chain Models of the Weather

Example 4.4 is tricky. We want to construct a Markov chain model. Hence, we must construct a square probability transition matrix. The idea is to make the state by the weather on two consecutive days. The state at step n is $X_n \equiv (w_{n-1}, w_n)$, where w_n is the type of weather on day n. The next state is $X_{n+1} \equiv (w_n, w_{n+1})$, the weather on days n and $n+1$. Note that the weather at day n, w_n, appears in *both* states. We can go from (D, R) to (R, D), but we cannot go from (D, R) to (D, R). Thus half the entries of the transition matrix must be 0. We can solve $\pi = \pi P$ to find the stationary probability vector, but this is a 1×4 probability vector. To get the long-run proportion of dray days, we need to add the probabilities for the two states (D, R) and (D, D) (or, equivalently, (R, D) and (D, D) focusing on the next day).