# IEOR 3106: Introduction to Operations Research: Stochastic Models

## Fall 2012, Professor Whitt

## Martingales and Gambling, Thursday, December 6

**1. Sources.** Martingales appear only briefly in the textbook, being introduced before Exercise 10.16 and discussed in the following problems. Some relevant material also appears in Exercises 7.13-7.15. Thus you should see the various on-line lecture notes. The 8-page Wikipedia entry is excellent as well.

**2. MG's as Generalizations of Random Walks.** A martingale (MG) is a generalization of a mean-zero random walk. A *simple random walk* goes up or down 1 at each step. That is, there is a sequence of i.i.d. Bernoulli random variables (taking values either $+1$ or $-1$) $\{X_n : n \geq 1\}$ with

$$P(X_n = +1) = p = 1 - P(X_n = -1). \tag{1}$$

If $p = 1/2$, then the simple random walk is a mean-zero random walk (with zero drift) or a *symmetric simple random walk*. Then we form the associated sequence of partial sums, letting $S_0 \equiv 0$ and

$$S_n \equiv X_1 + \cdots + X_n, \quad n \geq 1. \tag{2}$$

The process $\{S_n : n \geq 0\}$ is the simple random walk.

It is easy to see that the conditional expectation given the process history of a mean-zero simple random walk satisfies the key MG property

$$E[S_{n+1}|S_1, \ldots S_n] = S_n + E[X_n + 1|S_1, \ldots S_n] = S_n + E[X_n + 1] = S_n. \tag{3}$$

A martingale generalizes this property. First note that property remains true if the steps $X_n$ have an arbitrary distribution with $E[|X_n|] < \infty$ and $E[X_n] = 0$. If definition (2) holds when $\{X_n : n \geq 1\}$ is a sequence of i.i.d. random variables where each has an arbitrary distribution, then the process $\{S_n : n \geq 0\}$ is a general random walk or just a *random walk*. If $E[X_n] = 0$, then we have a mean-zero random walk, which also is a MG. The general mean-zero random walk is the canonical example of a MG. It is the discrete-time analog of Brownian motion. Or, expressed differently, Brownian motion is the continuous time analog of a random walk.

Moreover, the random variables $X_n$ need not be identically distributed. The random variables $X_n$ being independent with mean zero is enough to give a MG. But there are many other examples of MG's as well. We illustrate with a gambling example.

**3. Models of Gambling.**

The supplementary notes give a model for the evolution of a gambler's wealth when gambling in a sequence of games. The model in the notes can be generalized to allow multiple gambles (betting opportunities) in each time period. The main point is that if the games are all *fair* (have expected net payoff zero), then the wealth process in a MG.

A frequently used result is the MG Optional Stopping Theorem (OST), which shows that the expected value at a random stopping time is the same as the initial expected value (under conditions!!) The practical meaning is that *there is no free lunch.* You cannot get something out of nothing.

Various classical gambling systems seem to defy the Optional Stopping Theorem, but they highlight that the boundedness conditions are violated. In order to implement the strategy successfully, you need infinite wealth. See the other notes.

Martingales play a key role in probability theory and finance. This is the beginning of a big story.