IEOR 3106, Fall 2012, Professor Whitt

Professor Whitt, Thursday, December 6

Oatpower

Problem 1 on the 2011 final exam:

1. Oatpower, Inc. (30 points)

Ever since the 2003 power outage in the northeastern United States, there has been growing investor enthusiasm for the company Oatpower, Inc., which is developing a new way to efficiently generate vast power from ordinary oats. Oatpower claims that it will be possible to generate sufficient power from a single cup of oats to run a subway train for ten years. If Oatpower is successful, subways and elevators will no longer have to depend on America's aging electric power grid. The power generation method is highly secret, but there is a rumor that it is based on a surprising chemical reaction between oats and Raspberry Snapple.

The current price of Oatpower stock is \$100 per share. Suppose that the Oatpower stock price over time (measured in years) can be modelled as the stochastic process $\{S(t) : t \geq 0\}$, where

$$S(t) \equiv 100 + 5B(t), \quad t \geq 0,$$

and $\{B(t) : t \geq 0\}$ is standard (drift zero, unit variance) Brownian motion.

(a) (4 points) Calculate $E[S(4)]$ and $E[S(4)^2]$.

(b) (4 points) Calculate $P(S(4) > 110)$.

(c) (5 points) Let T_s be the first time that the stock price reaches the level s. Calculate $P(T_{110} \leq 4)$.

(d) (5 points) Let $T \equiv \min\{T_{90}, T_{140}\}$. Calculate $E[S(T)]$ and $E[T]$.

(e) (5 points) Calculate $P(T_{90} < T_{140} < T_{80})$.

(f) (3 points) Calculate $E[S(1)|S(4) = 120]$.

(g) (4 points) Calculate $E[S(1)^2|S(4) = 120]$.

Solutions

(a) (4 points) Calculate $E[S(4)]$ and $E[S(4)^2]$.

This problem is about Chapter 10.

$$E[S(4)] = E[100 + 5B(4)] = 100 + 5E[B(4)] = 100$$

because $E[B(t)] = 0$ for all t. Since $Var(a + bX) = b^2 Var(X)$ for any random variable X,

$$Var(S(4)) = Var(5B(4)) = 25Var(B(4)) = 25 \times 4 = 100.$$

Then the second moment is

$$E[S(4)^2] = Var(S(4)) + E[S(4)]^2 = 100 + (100)^2 = 10,100$$

(b) (4 points) Calculate $P(S(4) > 110)$.

From part (a), $S(4)$ is distributed as $N(100, 100)$. Hence,

$$\begin{aligned} P(S(4) > 110) &= P(N(100,100) > 110) = P(100 + 10N(0,1) > 110) \\ &= P(N(0,1) > 1) \approx 0.16 \end{aligned}$$

by the table on p. 82.

(c) (5 points) Let T_s be the first time that the stock price reaches the level s. Calculate $P(T_{110} \leq 4)$.

$$P(T_{110} \leq 4) = P(\max_{0 \leq t \leq 4} \{S(t)\} > 110) = 2P(S(4) > 110) = 2(0.16) = 0.32$$

by §10.2 of the book and then by part (a).

(d) (5 points) Let $T \equiv \min\{T_{90}, T_{140}\}$. Calculate $E[S(T)]$ and $E[T]$.

It is helpful to rephrase the question in terms of ordinary Brownian motion. The hitting time T is distributed the same as $T \equiv \min\{T_{-2}, T_8\}$ for ordinary Brownian motion. That is, the original stochastic process $S(t)$ hits either 90 or 140 the same time that the component $B(t)$ hits either -2 or $+8$.

We use the optional stopping theorem with martingales to obtain

$$E[B(T)] = E[B(0)] = 0, \quad \text{so that} \quad E[S(T)] = E[S(0)] = 100.$$

For the second part we can However, by Exercise 10.18 and by the lecture notes, $B(t)^2 - t$ is a martingale so that

$$E[T] = 2 \times 8 = 16$$

(e) (5 points) Calculate $P(T_{90} < T_{140} < T_{80})$.

This is just like Exercise 10.5. First, since we start at $S(0) = 100$, we use $E[S(T)] = 0$ to obtain

$$P(T_{90} < T_{140}) = \frac{4}{4+1} = \frac{4}{5}.$$

After we hit 90, we have a second independent problem of hitting 140 before 80.

$$P(T_{140} < T_{80} | T_{90} < T_{140}) = \frac{1}{1+5} = \frac{1}{6}.$$

Since these two events are independent,

$$P(T_{90} < T_{140} < T_{80}) = \left(\frac{4}{5}\right) \times \left(\frac{1}{6}\right) = \frac{2}{15}$$

(f) (3 points) Calculate $E[S(1)|S(4) = 120]$.

Here you should use the first equation in display (10.4) in §10.1.

$$E[S(1)|S(4) = 120] = 100 + \frac{1}{4}20 = 105$$

Again, it may be helpful to rephrase the question in terms of ordinary Brownian motion.

$$E[S(1)|S(4) = 120] = 100 + 5E[B(1)|B(4) = 4] = 100 + 5 \times 1 = 105.$$

(g) (4 points) Calculate $E[S(1)^2|S(4) = 120]$.

Here we should use the first equation in display (10.4) in §10.1.

$$Var(S(1)|S(4) = 120) = \left(\frac{1 \times 3}{4}\right) 25 = \frac{75}{4}.$$

Again, it may be helpful to rephrase the question in terms of ordinary Brownian motion.

$$Var(S(1)|S(4) = 120) = 25Var(B(1)|B(4) = 4) = 25 \left(\frac{1 \times 3}{4}\right) = \frac{75}{4}.$$

Hence, the second moment is

$$E[S(1)^2|S(4) = 120] = Var(S(1)|S(4) = 120) + (E[S(1)|S(4) = 120])^2 = \frac{75}{4} + (105)^2 = 11,043.75$$

It would be OK to omit the final calculation.
