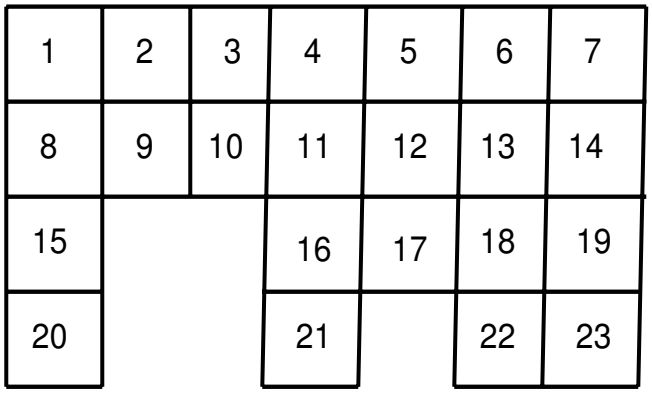
IEOR 3106: Introduction to Operations Research: Stochastic Models

Part 2 of First Midterm Exam, Chapter 4, October 7, 2008

You moose show your work.

Markov Moose's Habitat



1. Markov Moose Moving In His Habitat (15 points)

Markov Moose is a creature of habit; he moves around in a fixed (irregular) habitat. But Markov Moose is also a random creature of habit; he moves around randomly in a Markovian manner. To describe Markov Moose's movements, we divide his irregular habitat, depicted in the figure above, into 23 square regions. We monitor the successive regions that Markov Moose visits (without paying attention to the amount of time spent in each region during each visit). Let X_0 be the initial region occupied by Markov Moose, and let X_n be the region occupied by Markov Moose after n transitions. Assuming that Markov Moose moves around in an aimless manner, we assume that, in each transition, he moves to one of the allowable neighboring regions (vertically or horizontally), each with equal probability. (From region 1, he moves to either region 2 or region 8, each with probability 1/2; from region 20, he move to region 15 with probability 1.) The following questions concern the stochastic process $\{X_n : n \geq 0\}$.

(a) (3 points) What is the long-run proportion of transitions in which Markov Moose ends up in region 1?

(b) (3 points) Given that Markov Moose starts in region 1, what is the probability that he is in region 1 after 2 transitions?

(c) (3 points) Given that Markov Moose starts in region 1, what is the (approximate) probability that he is in region 1 after 231 transitions?

(d) (3 points) Given that Markov Moose starts in region 1, what is the (approximate) probability that he is in region 2 after 231 transitions?

(e) (3 points) Given that Markov Moose starts in region 1, what is the expected number of transitions before he is first back in region 1?

2. Is Markov Moose mellow? (15 points)

On any given day, Markov Moose can be either angry or mellow (peaceful). (You do not want to meet an angry moose, even in a car, with either him at the wheel or you.) Upon close examination, it has been observed that "one out of three mellow days is followed by an angry day, but only one out of six angry days is followed by a mellow day."

(a) (2 points) Construct a Markov chain model of Markov Moose's mood over successive days, consistent with the observation above.

(b) (2 points) Does the observation above imply that a Markov chain model is necessarily appropriate? Explain.

For the next three parts, use your Markov chain model constructed in part (a).

(c) (2 points) Given that Markov Moose is initially mellow, what is the probability that Markov Moose is first angry six days later?

(d) (2 points) Given that Markov Moose is initially mellow, what is the (approximate) probability that Markov Moose is mellow 30 days later?

(e) (2 points) Given that Markov Moose is initially mellow, what is the expected number of days until he is first mellow again?

(f) (3 points) Suppose that we seek a better model of Markov Moose's mood over successive days. To do so, we exploit knowledge of his mood on two consecutive days. Toward that end, suppose that three out of four times that Markov Moose's mood is mellow on both days $n-1$ and n it is also mellow on day $n+1$; one out of two times that Markov Moose's mood is angry on day $n-1$ but mellow on day n it is also mellow on day $n+1$; one out of five times that Markov Moose's mood is mellow on day $n-1$ but angy on day n it is mellow on day $n+1$; one out of seven times that Markov Moose's mood is angry on both days $n-1$ and n it is mellow on day $n+1$. Construct a Markov chain model of Markov Moose's mood over successive days, consistent with these new observations.

(g) (2 points) Is the Markov chain you constructed in part (f) a periodic Markov chain? Explain.

3. A Markov Chain Transition Matrix (20 points)

Consider a Markov chain on the ten states $\{1, 2, \cdots, 10\}$ with transition matrix P given by

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{pmatrix} 0.3 & 0.1 & 0.0 & 0.0 & 0.0 & 0.1 & 0.2 & 0.2 & 0.0 & 0.1 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.3 & 0.0 & 0.0 & 0.0 & 0.7 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.7 & 0.0 & 0.0 & 0.0 & 0.3 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.1 & 0.0 & 0.0 & 0.1 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.2 & 0.8 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{pmatrix}$$

**Note that we are numbering the states 1, 2, ... , 10, with the columns numbered in the same order as the rows.

Please answer the following questions. Two points are subtracted for each wrong answer, up to 20 points. However, two bonus positive points are given for correctly answering each of the last two parts.

(a) Which states are accessible from state 1?

(b) From which states is state 1 accessible?

(c) Do states 1 and 6 communicate?

(d) Identify the communication classes for this Markov chain.

(e) Which communication classes are closed? Which are open?

(f) Which states are transient? Which states are recurrent?

(g) Put the transition matrix in canonical form.

In the following questions, we are referring to the states as originally defined and numbered.

(h) Compute the six-step transition probability $P_{2,7}^{(6)}$.

(i) Compute the two-step transition probability $P_{4,10}^{(2)}$.

(j) Compute the two-step transition probability $P_{1,2}^{(2)}$.

(k) Starting in state 3, what is the expected total number of visits to state 7?

(l) Starting in state 1, what is the expected total number of visits to state 5?

(m) (+2 bonus points) Starting in state 1, what is the expected total number of visits to state 8?

(n) (+2 bonus points) Starting in state 1, what is the probability of ever visiting state 5?

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing "I have neither given not received improper help on this examination," on your examination booklet and sign your name. You may keep the exam itself. Solutions will eventually be posted on line.