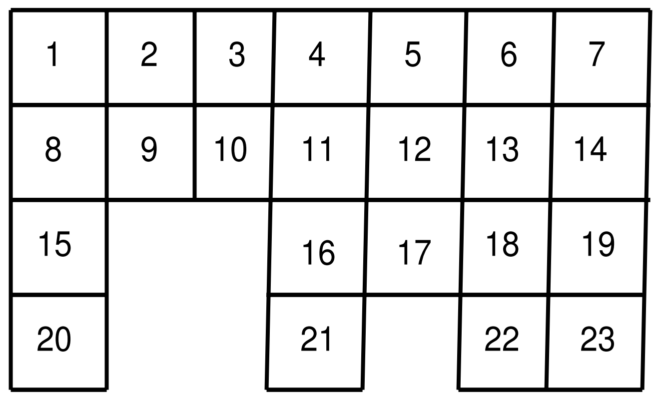
IEOR 3106: Introduction to Operations Research: Stochastic Models

SOLUTIONS to Part 2 of First Midterm Exam, October 7, 2008

You moose show your work.

Markov Moose's Habitat



1. Markov Moose Moving In His Habitat (15 points)

Markov Moose is a creature of habit; he moves around in a fixed (irregular) habitat. But Markov Moose is also a random creature of habit; he moves around randomly in a Markovian manner. To describe Markov Moose's movements, we divide his irregular habitat, depicted in the figure above, into 23 square regions. We monitor the successive regions that Markov Moose visits (without paying attention to the amount of time spent in each region during each visit). Let X_0 be the initial region occupied by Markov Moose, and let X_n be the region occupied by Markov Moose after n transitions. Assuming that Markov Moose moves around in an aimless manner, we assume that, in each transition, he moves to one of the allowable neighboring regions (vertically or horizontally), each with equal probability. (From region 1, he moves to either region 2 or region 8, each with probability 1/2; from region 20, he move to region 15 with probability 1.) The following questions concern the stochastic process $\{X_n : n \geq 0\}$.

(a) (3 points) What is the long-run proportion of transitions in which Markov Moose ends up in region 1?

We just want to solve for the steady-state probability, which can be done by solving $\pi = \pi P$, but here the transition matrix P is 23×23, so we would have to solve a system of 23 linear equations. But we do not need to do that, because this Markov chain has special structure, just like the Markov mouse moving in a closed maze. That is the structure of a **random walk on a weighted graph**. For such a model, we can exploit reversibility, as discussed in Section 4.8 of Ross. That gives a simple answer: We can count the number of ways out of each region. We divide the number of ways out of region 1 by the sum of the numbers of ways over all regions. (Each "doorway" thus gets counted twice.) Observe that there are 2 regions with only 1 way out (regions 20 and 21); there are 5 regions with 2 ways out (regions 1, 7, 15, 22 and 23); there are 12 regions with 3 ways out (regions 2, 3, 4, 5, 6, 8, 9, 10, 14, 16, 17 and 19); and there are 4 regions with 4 ways out (regions 11, 12, 13 and 18). There are

$2+5+12+4=23$ regions in all.

Hence the long-run proportion of transitions spent in region 1 is $2/64 = 1/32$.

(b) (3 points) Given that Markov Moose starts in region 1, what is the probability that he is in region 1 after 2 transitions?

$$P_{1,1}^{(2)} = P_{1,2}P_{2,1} + P_{1,8}P_{8,1} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{3}.$$

(c) (3 points) Given that Markov Moose starts in region 1, what is the (approximate) probability that he is in region 1 after 231 transitions?

As with Markov mouse discussed in class, this chain has period two, as can be seen by coloring alternating squares red and black. The chain moves from red squares to black squares, and then from black squares to red squares (but not from even to odd or odd to even as before). Hence, the chain can go from state 1 to state 1 only in an even number of steps. So the **answer** is 0.

(d) (3 points) Given that Markov Moose starts in region 1, what is the (approximate) probability that he is in region 2 after 231 transitions?

We can apply the steady-state probability of being in region 2, because that will be positive in an odd number of steps. Because the chain is periodic, with period 2, we have to multiply the steady-state probability times 2. The steady state probability of being in region 2 is (ways out of 2)/sum of ways out of all rooms), which is $3/64$ (see part (a)). Hence, the **answer** is $2 \times (3/64) = 6/64 = 3/32$.

(e) (3 points) Given that Markov Moose starts in region 1, what is the expected number of transitions before he is first back in region 1?

Apply part (a). The mean time to return to region 1, starting from region 1 is the reciprocal of the long-run proportion: $1/(1/32) = 32$ transitions. See Remark (ii) on page 212 of Ross.

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2. Is Markov Moose mellow? (15 points)

On any given day, Markov Moose can be either angry or mellow (peaceful). (You do not want to meet an angry moose, even in a car, with either him at the wheel or you.) Upon close examination, it has been observed that "one out of three mellow days is followed by an angry day, but only one out of six angry days is followed by a mellow day."

(a) (2 points) Construct a Markov chain model of Markov Moose's mood over successive days, consistent with the observation above.

(a) Please note that this is a minor variation of the "cars-and-trucks" problem - Homework Exercise 4.30, with the state names changed in order to confuse and befuddle you. Hopefully you saw through this subterfuge. As usual, the Markov chain model is defined by defining the Markov chain transition matrix P. Here the transition matrix should be:

$$P = \begin{matrix} M \\ A \end{matrix} \begin{pmatrix} 2/3 & 1/3 \\ 1/6 & 5/6 \end{pmatrix} .$$

The columns are understood to be labelled in the same way as the rows. You could have ordered the two states in the other way.

(b) (2 points) Does the observation above imply that a Markov chain model is necessarily appropriate? Explain.

No, as discussed in class, the original observation does not directly imply the Markov property. The Markov property states that the conditional probability of a future state, given the present state and any past states, depends only on the present state. That is, formally,

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, \ldots, X_1 = i_1) = P(X_{n+1} = j | X_n = i)$$

for all n, j, i, and past states. In class we demonstrated that by presenting an alternative deterministic model, which is consistent with the statement, and yet does not satisfy the Markov property. The idea is to present a deterministic repetitive pattern. For this problem, the pattern should be $(M, M, M, A, A, A, A, A, A)$. This pattern is repeated forever; i.e., the successive days can be:

$$M, M, M, A, A, A, A, A, A, M, M, M, A, A, A, A, A, A, M, M, M, A, A, A, A, A, A, \ldots$$

Note that the pattern of length 9 is repeated indefinitely. Note that 1 of 3 mellow days (M) is followed by an angry day (A), but only 1 out of 6 angry days is followed by a mellow day. (The last A in each pattern of length 9 is followed by the M in the first place in the next pattern.)

For the next three parts, use your Markov chain model constructed in part (a).

(c) (2 points) Given that Markov Moose is initially mellow, what is the probability that Markov Moose is first angry six days later?

$$(2/3)^5(1/3) = (2)^5/(3)^6 = 32/729.$$

(d) (2 points) Given that Markov Moose is initially mellow, what is the (approximate) probability that Markov Moose is mellow 30 days later?

Here we solve $\pi = \pi P$ to get $\pi = (1/3, 2/3)$. We use $\pi_M = 1/3$ as an approximation.

(e) (2 points) Given that Markov Moose is initially mellow, what is the expected number of days until he is first mellow again?

Apply part (d). The mean time to return to mellowness, starting from a mellow state is the reciprocal of the long-run proportion: $1/(1/3) = 3$ days. Again see Remark (ii) on page 212 of Ross.

(f) (3 points) Suppose that we seek a better model of Markov Moose's mood over successive days. To do so, we exploit knowledge of his mood on two consecutive days. Toward that end, suppose that three out of four times that Markov Moose's mood is mellow on both days $n-1$ and n it is also mellow on day $n+1$; one out of two times that Markov Moose's mood is angry on day $n-1$ but mellow on day n it is also mellow on day $n+1$; one out of five times that Markov Moose's mood is mellow on day $n-1$ but angy on day n it is mellow on day $n+1$; one out of seven times that Markov Moose's mood is angry on both days $n-1$ and n it is mellow on day $n+1$. Construct a Markov chain model of Markov Moose's mood over successive days, consistent with these new observations.

This is just like Example 4.4 on p. 186 of Ross. We let the state be the pair (X_{n-1}, X_n). The next state is the pair (X_n, X_{n+1}). Not that X_n appears as the second day in the first pair and in the first day of the second pair. Necessarily they must agree; otherwise the transition probability must be 0. There are now four states. The new transition matrix is

$$P = \begin{matrix} MM \\ AM \\ MA \\ AA \end{matrix} \begin{pmatrix} 3/4 & 0 & 1/4 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/5 & 0 & 4/5 \\ 0 & 1/7 & 0 & 6/7 \end{pmatrix} .$$

(g) (2 points) Is the Markov chain you constructed in part (f) a periodic Markov chain? Explain.

No, it is not periodic. That is easy to see, because $P_{MM,MM} > 0$; i.e., it is possible to go from the state MM back to itself in 1 step, and thus in any finite number of steps.

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3. A Markov Chain Transition Matrix (20 points)

Consider a Markov chain on the ten states $\{1, 2, \cdots, 10\}$ with transition matrix P given by

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{pmatrix} 0.3 & 0.1 & 0.0 & 0.0 & 0.0 & 0.1 & 0.2 & 0.2 & 0.0 & 0.1 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.3 & 0.0 & 0.0 & 0.0 & 0.7 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.7 & 0.0 & 0.0 & 0.0 & 0.3 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.1 & 0.0 & 0.0 & 0.1 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.2 & 0.8 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{pmatrix}$$

**Note that we are numbering the states 1, 2, ... , 10, with the columns numbered in the same order as the rows.

Please answer the following questions. Two points are subtracted for each wrong answer, up to 20 points. However, two bonus positive points are given for correctly answering each of the last two parts.

(a) Which states are accessible from state 1?

Remember that j being *accessible* from i means that you can get from i to j in some finite number of steps, not necessarily in a single step. It turns out that **all 10 states** are accessible from state 1. This can be determined by constructing a graph showing the $1-step$ connectivity. It becomes clear from the canonical form of the transition matrix, in part (g). To answer this part, you should be beginning your construction of that canonical form. You could do part (g) first.

(b) From which states is state 1 accessible?

The principle is the same, but it is easier to leave state 1 than to get to it. In fact, state 1 is accessible only from states 1 and 8.

(c) Do states 1 and 6 communicate?

No, states 1 and 6 do *not* communicate. They communicate if each is accessible from the other. That is not true. State 6 is accessible from state 1, but state 1 is not accessible from state6.

(d) Identify the communication classes for this Markov chain.

Each communication class is a subset of the states. When we form communication classes, we construct a *partition* of the set of states (into disjoint subsets whose union is the whole set). Here there are 5 communication classes: $\{2\}$, $\{3,7\}$, $\{4,10\}$, $\{5,6,9\}$, and $\{1,8\}$.

(e) Which communication classes are closed? Which are open?

The closed classes are the classes that you cannot leave. The open classes are the classes from which you can leave, and thus eventually will leave with probability 1. There are four closed classes: $\{2\}$, $\{3,7\}$, $\{4,10\}$ and $\{5,6,9\}$. There is one open class: $\{1,8\}$.

(f) Which states are transient? Which states are recurrent?

The states in closed communication classes are recurrent: You will return with probability 1. The states in open communication classes are transient: You will eventually leave for the last time and never return again after that. The transient states are 1 and 8; the others are recurrent.

(g) Put the transition matrix in canonical form.

Here is the canonical form of the transition matrix P:

$$P = \begin{array}{c} 2 \\ 3 \\ 7 \\ 4 \\ 10 \\ 5 \\ 6 \\ 9 \\ 1 \\ 8 \end{array} \left(\begin{array}{cccccccccc} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.7 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.7 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.2 & 0.8 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.0 & 0.2 & 0.0 & 0.1 & 0.0 & 0.1 & 0.0 & 0.3 & 0.2 \\ 0.1 & 0.0 & 0.2 & 0.0 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{array} \right)$$

We have re-ordered the states so that the 4 closed communication classes appear together at the top, while the single open communication class appears together at the bottom. The original states are listed on the left. As before, the columns and rows both have this same ordering. For example, the entry 1.0 in the upper left corner is the transition probability $P_{2,2}$, because state 2 has been moved to being first. We have ordered the four closed communication classes by size, putting the smaller ones first, but that is optional. You could have a different matrix, but the states in the same communication class must appear together, next to each other, and the transient states must appear at the bottom.

In the following questions, we are referring to the states as originally defined and numbered.

(h) Compute the six-step transition probability $P_{2,7}^{(6)}$.

Note that state 7 is not accessible from state 2, so this is a "freebie." Here $P_{2,7}^{(6)} = 0$.

(i) Compute the two-step transition probability $P^{(2)}_{4,10}$.

Note that states 4 and 10 belong to the same communicating class, so here there is something to compute. It suffices to look at the little 2×2 sub-matrix, which is a transition matrix in its own right. We have

$$P = \begin{matrix} 4 \\ 10 \end{matrix} \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{pmatrix} .$$

The columns are understood to be labelled in the same way as the rows. To proceed, we calculate the two-step transition matrix $P^{(2)} = P^2$ for this little DTMC. We get

$$P^2 = \begin{matrix} 4 \\ 10 \end{matrix} \begin{pmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{pmatrix} .$$

Hence, $P^{(2)}_{4,10} = 0.44$.

(j) Compute the two-step transition probability $P^{(2)}_{1,2}$.

Things are a little more complicated here, because 1 is a transient state, while 2 is an absorbing state. But you just need to think clearly: There are *three* possibilities, i.e.,

$$P^{(2)}_{1,2} = P_{1,2}P_{2,2} + P_{1,1}P_{1,2} + P_{1,8}P_{8,2} = (0.1)(1.0) + (0.3)(0.1) + (0.2)(0.1) = 0.15$$

(k) Starting in state 3, what is the expected total number of visits to state 7?

State 3 is a recurrent state, belonging to the same closed communicating class with state 7. Thus the expected total number of visits to state 7, starting in state 3, is infinite.

(l) Starting in state 1, what is the expected total number of visits to state 5?

This might at first look complicated, but you only need remember that infinity times any positive number is infinity, and infinity plus any finite number is infinity. There is a positive probability of going from state 1 to state 10 in one step; once in state 10, the expected number of visits to state 5 is infinite. So the expected number of visits to state 5 starting in state 1 is infinite. This is true even though we cannot get from state 1 to state 5 in one step, and there is a significant chance (probability) that we will *never* reach state 5 from state 1.

(m) (+2 bonus points) Starting in state 1, what is the expected total number of visits to state 8?

Finally, we have come to a question that requires some work. These last two parts require that you know how to work with the fundamental matrix of an absorbing chain $N = (I - Q)^{-1}$. Even though the entire chain is not an absorbing chain, we can analyze movement among the two transient states as if it were. Here then we focus on the sub-matrix Q corresponding to thew two transient states 1 and 8. We have

$$Q = \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 0.3 & 0.2 \\ 0.1 & 0.1 \end{pmatrix} .$$

The columns are understood to be labelled in the same way as the rows. Then we have

$$I - Q = \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 0.7 & -0.2 \\ -0.1 & 0.9 \end{pmatrix}$$

Now, for the first time, you have to do some mathematics; you need to invert a 2×2 matrix, but even if you cannot do that, perhaps because you ran out of time, you could at least quickly write own the formula. You want $N_{1,8}$ where $N = (I - Q)^{-1}$. All that is left is the linear algebra: The answer is

$$N = (I - Q)^{-1} = \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 90/61 & 20/61 \\ 10/61 & 70/61 \end{pmatrix} .$$

Therefore, the **answer** is $N_{1,8} = 20/61$.

You could easily mess up in this step, but it is easy to check your answer; you should have $N \times (I - Q) = I$. That is easy to check. Because of the probabilistic interpretation (and the associated mathematical structure), the elements of N must all be nonnegative. Here is the gruesome analysis in detail: The successive operations on the matrix $I - Q$ look like this:

$$I - Q = \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 0.7 & -0.2 \\ -0.1 & 0.9 \end{pmatrix}$$

$$\text{step 1:} \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 1 & -2/7 \\ -0.1 & 0.9 \end{pmatrix}$$

$$\text{step 2:} \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 1 & -2/7 \\ 0 & 61/70 \end{pmatrix}$$

$$\text{step 3:} \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 1 & -2/7 \\ 0 & 1 \end{pmatrix}$$

$$\text{step 4: } I = \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The corresponding operations applied to the identity matrix are:

$$I = \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{step 1:} \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 10/7 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{step 2:} \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 10/7 & 0 \\ 1/7 & 1 \end{pmatrix}$$

$$\text{step 3:} \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 10/7 & 0 \\ 10/61 & 70/61 \end{pmatrix}$$

$$\text{step 4: } (I - Q)^{-1} = \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 90/61 & 20/61 \\ 10/61 & 70/61 \end{pmatrix}$$

(n) (+2 bonus points) Starting in state 1, what is the probability of ever visiting state 5?

And here is yet another question that requires some work, but not too much if you know what you are doing. You can exploit the previous part. Note that state 1 is a transient state, while 5 is one of the recurrent states that we might get to from state 1. We need to compute the probability of being absorbed in the closed communicating class containing state 5, starting from state 1. We first collapse the closed communicating classes to single states in order to construct an absorbing Markov chain. We replace the closed communicating classes by single states. For the transitions from transient to communicating classes, we must appropriately add the values.

The new reduced transition matrix has 6 states: one for each of the 4 closed communicating classes and 2 for the 2 transient states. The new reduced absorbing transition matrix is:

$$P = \begin{matrix} 2 \\ \{3,7\} \\ \{4,10\} \\ \{5,6,9\} \\ 1 \\ 8 \end{matrix} \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.1 & 0.1 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.3 & 0.1 & 0.1 \end{pmatrix}$$

As discussed in the lecture notes of September 16, here the absorbing DTMC has the general form:

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} ,$$

where I is an identity matrix (1's on the diagonal and 0's elsewhere) and 0 (zero) is a matrix of zeros. In this case, I would be 4×4, R is 2×4 and Q is 2×2. The matrix Q describes the probabilities of motion among the transient states, as already discussed. The matrix R gives the probabilities of absorption in one step (going from one of the transient states to one of the absorbing states in a single step). Here the absorbing states correspond to entire communicating classes in the original DTMC with 10 states. In general Q would be square, say m by m, while R would be m by k, and I would be k by k.

In this framework, we want to compute the matrix $B = NR$, using the fundamental matrix N already computed in the previous part (n). Since 1 is the 5^{th} state in this new matrix and since the target destination state 5 belongs to the fourth communication class, we want to compute $B_{5,4}$.

Here

$$B = NR = \begin{matrix} 1 \\ 8 \end{matrix} \begin{pmatrix} 11/61 & 22/61 & 13/61 & 15/61 \\ 8/61 & 16/61 & 15/61 & 22/61 \end{pmatrix}$$

As a check, note that the entries must be probabilities. The entries are nonnegative and the row sums are 1. We want $B_{1,4}$, so that the **answer** is $= 15/61$.

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