IEOR 3106: Introduction to Operations Research: Stochastic Models

SOLUTIONS to Part I of First Midterm Exam, September 30, 2008

You moose show your work.

Figure 1: a moose: Alces alces

From Wikipedia, the free encyclopedia: The moose (North America) or elk (Europe), Alces alces, is the largest extant species of the deer family. Moose are distinguished by the palmate antlers of the males, as depicted in Figure 1; other members of the deer family have antlers with a "twig-like" configuration.

1. The Moose-Recognition Test. (15 points)

In an animal-recognition test, 99% of Alaskan high school students can properly identify a moose from a picture, but only 10% of other American high school students can properly identify a moose from a picture. Alaskan high school students make up approximately 2% of the American high-school student population.

(a) (5 points) What is the probability that a randomly selected American high school student can properly identify a moose from a picture?

It is good to draw a picture. Here the relevant picture is a probability tree, as in the figure below. Let "rec" stand for properly recognizing or identifying the moose. Add the probabilities for the two "rec" ends to get

$$P(\text{identify}) = 0.0198 + 0.0980 = 0.1178$$

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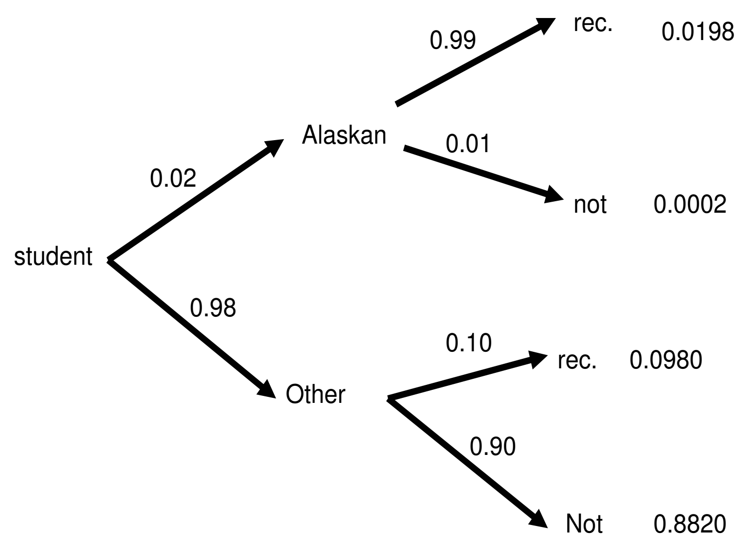
probability tree for moose recognition test



(b) (10 points) What is the conditional probability that a randomly selected American high school student is from Alaska, given that the student can properly identify a moose from a picture?

This is the quintessential problem from Chapter 1. See Sections 1.4-1.6.

$$P(\text{Alaskan}|\text{identify}) = \frac{P(\text{Alaskan} \bigcap \text{identify})}{P(\text{identify})} = \frac{0.0198}{0.1178} = 0.1681$$

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2. The Great Moose Weigh-Off. (15 points)

The male moose is an impressively large animal. The male moose is so big that it has been claimed that 100 randomly selected male moose together would weigh more than a diesel locomotive (train engine). A mayor of a small Alaskan town has decided to test that claim by having *The Great Moose Weigh-Off*. Suppose that the selected diesel locomotive weighs $52,000$ kilograms (kgs.). Suppose that a random male moose has a weight that is distributed according to a gamma distribution, having mean 500 kgs. and standard deviation 100 kgs.

(a) (10 points) Under these assumptions, what is the approximate probability that 100 randomly and independently selected male moose together weigh more than the selected diesel locomotive?

Let X_k be the weight of the k^{th} moose. Let $S_n \equiv X_1 + \cdots + X_n$ be the sum of the weights of the first n moose. We do not need to use the gamma distribution assumption. That was a red herring. By the central limit theorem, S_{100} is approximately normally distributed. First we have $E[X_1] = 500$ by assumption and $Var(X_1) = (100)^2 = 10,000$. Then $E[S_{100}] = 50,000$ and $Var(S_{100}) = 100 * Var(X_1) = 1,000,000$, so that the standard deviation of S_{100} is $\sqrt{Var(S_{100})} = 1,000$. Hence,

$$\begin{aligned} P(S_{100} > 52,000) &= P\left(\frac{S_{100} - E[S_{100}]}{SD(S_{100})} > \frac{52,000 - E[S_{100}]}{SD(S_{100})}\right) \\ &\approx P\left(N(0,1) > \frac{52,000 - E[S_{100}]}{SD(S_{100})}\right) \\ &\approx P\left(N(0,1) > \frac{52,000 - 50,000]}{1000}\right) \\ &\approx P(N(0,1) > 2) \approx 0.0228 \end{aligned}$$

from the table on page 81.

(b) (5 points) Explain the mathematical basis for your answer in part (a).

The mathematical basis is the central limit theorem. That tells us the distribution is approximately normal, regardless of the distribution of X_1, provided that the distribution of X_1 has finite mean and variance, which has been assumed. See page 79 of the text. See the lectures notes of Tuesday, September 9. For a proof, see the last pages of the notes of September 11.

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3. A Moose Habitat (20 points)

An old grey moose has settled down to live in a comfortable swampy habitat, which happens to occupy a one-mile by one-mile square region, as depicted in the figure below. There is a salt lick in the northeast corner of the habitat, which the moose likes to visit.

Let the random vector (X, Y) represent the random location of the old grey moose at any time. Let the random vector have the probability density function

$$f_{X,Y}(x, y) = x + y, \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Determine the following quantities:

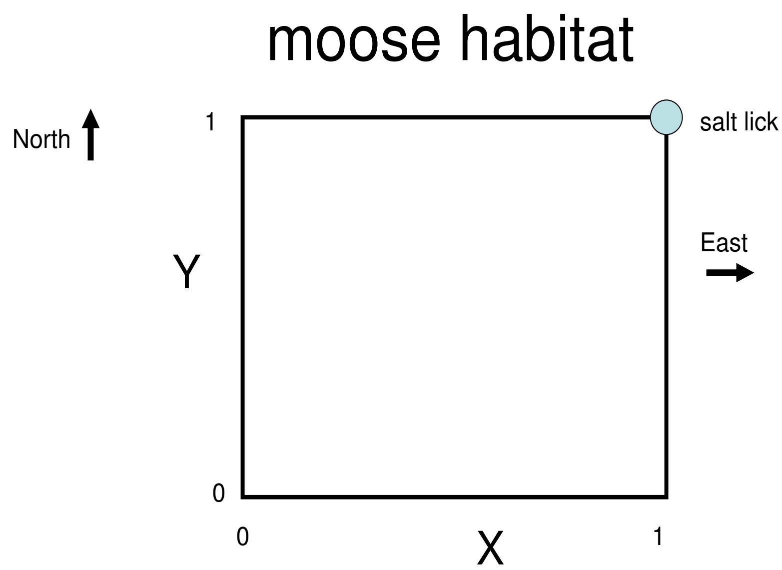
(a) (3 points) the probability density function of X,

$$f_X(x) = \int_0^1 f_{X,Y}(x, y)\, dy = \int_0^1 (x + y)\, dy = x + \frac{1}{2}$$

for $0 \le x \le 1$. To check, note that it is nonnegative and integrates to 1.

(b) (3 points) the cumulative distribution function of X

$$F_X(x) = \int_0^x f_X(u)\, du = \int_0^x (u + (1/2))\, du = (x^2 + x)/2, \quad 0 \le x \le 1.$$



To check, note that $F_X(0) = 0$ and $F_X(1) = 1$.

(c) (3 points) $E[X]$,

$$E[X] = \int_0^1 x f_X(x)\,dx = \int_0^1 (2x^2 + x)/2\,dx = (1/3) + (1/4) = 7/12.$$

(d) (3 points) $Var(X)$, the variance of X,

First,

$$E[X^2] = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 (2x^3 + x^2)/2\,dx = (1/4) + (1/6) = 5/12.$$

Then

$$Var(X) = E[X^2] - (E[X])^2 = (5/12) - (7/12)^2 = \frac{60 - 49}{144} = \frac{11}{144}.$$

(e) (4 points) $Cov(X, Y)$, the covariance of X and Y,

First

$$Cov(X, Y) = E[XY] - E[X]E[Y].$$

By symmetry, the distributions of X and Y are the same. Hence, $E[Y] = E[X] = 7/12$ by part (c). Then

$$\begin{aligned} E[XY] &= \int_0^1 \int_0^1 xy f_{X,Y}(x,y)\,dx\,dy \\ &= \int_0^1 \int_0^1 xy(x+y)\,dx\,dy \\ &= \int_0^1 \int_0^1 (x^2y + y^2x)\,dx\,dy \\ &= 2\int_0^1 \int_0^1 x^2y\,dx\,dy \\ &= 2\int_0^1 (x^2/2)\,dx = 1/3. \end{aligned} \tag{1}$$

Hence,

$$Cov(X,Y) = E[XY] - E[X]E[Y] = (1/3) - (7/12)^2 = \frac{-1}{144}.$$

Since the covariance is not 0, the two random variables X and Y are necessarily dependent. The random variables are slightly negatively correlated. That means bigger (smaller) X tends to imply smaller (bigger) Y.

(f) (4 points) $P(X \geq 1/2, Y \geq 1/2)$ (How likely is the moose to be near the salt lick?)

$$\begin{aligned} P(X \geq 1/2, Y \geq 1/2) &= \int_{1/2}^1 \int_{1/2}^1 f_{X,Y}(x,y)\,dx\,dy \\ &= \int_{1/2}^1 \int_{1/2}^1 (x+y)\,dx\,dy \\ &= \int_{1/2}^1 [(x+1/2) - ((x/2) + (1/8))]\,dx \\ &= \int_{1/2}^1 [(x/2) + 3/8)]\,dx \\ &= (5/8) - (1/4) = 3/8. \end{aligned} \tag{2}$$

If the four $(1/2)\times(1/2)$ regions ($[0,1/2]\times[0,1/2]$, $[0,1/2]\times[1/2,1]$, $[1/2,1]\times[0,1/2]$, $[1/2,1]\times[1/2,1]$) were equally likely, then the probability of each would be 1/4. We see that the moose is indeed somewhat more likely to be near the salt lick than if his location were uniformly distributed in the square habitat.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing "I have neither given not received improper help on this examination," on your examination booklet and sign your name. You may keep the exam itself. Solutions will eventually be posted on line.