IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2008, Professor Whitt, Second Midterm Exam

Chapters 5-6 in Ross, Tuesday, November 11

Open Book: but only the Ross textbook plus the CTMC Notes and one 8×11 page of notes

Justify your answers; show your work.

1. Defects in a Cable (45 points)

Suppose that minor defects are distributed over the length of a cable according to a Poisson process with rate 9 per mile and, independently, major defects are distributed over the same cable according to a Poisson process with rate 1 per mile. Suppose that we start examining a stretch of cable starting from a designated initial point, which we denote by $t = 0$.

[Note: Only the last part below, part (h), requires a numerical answer.]

(a) (5 points) What is the probability that there are exactly 3 defects, either minor or major, in the interval $[0, 1/2]$?

(b) (5 points) What is the conditional probability that there are exactly 3 *major* defects in the interval $[0, 1/2]$, given that there are 4 *minor* defects in the interval $[0, 1/2]$?

(c) (5 points) What is the probability that the second defect (after 0) is a major defect?

(d) (5 points) Let T_3 be the location of the third defect (after 0, either minor or major). What is $E[T_3]$?

(e) (5 points) What is the variance of T_3?

(f) (6 points) What is the probability that there are exactly 2 *major* defects in the interval $[0, 3]$, given that there are exactly 6 *major* defects in the interval $[0, 10]$?

(g) (7 points) What is the probability that there are exactly 2 *major* defects in the interval $[0, 3]$, given that there are exactly 6 defects of any kind, either major or minor, in the interval $[0, 10]$?

(h) (7 points) What is the approximate probability that there are more than 120 major defects in the interval $[0, 100]$? Is that probability greater than 0.05?

2. Two ATM Machines in Lerner Hall. (40 points)

There are two Automatic Teller Machines (ATM's) behind the small restaurant on the first floor of Lerner Hall. Suppose that customers arrive to receive service from one of these ATM machines according to a Poisson process with rate 2 per minute. Suppose that half of these arrivals (each with probability 1/2 independent of everything else) will elect to balk (leave immediately instead of joining the queue) if they cannot receive service immediately upon arrival. In addition, suppose that no arrival will stay if there are already 3 people waiting in addition to 2 being served. (We then say that the arrival is blocked.) Suppose that customer service times at these ATM machines are mutually independent exponential random variables with a mean of 1 minute. Suppose that each customer in queue has limited patience, and is willing to stay only a certain random amount of time before starting service, with the length of time depending upon the customer. Suppose that the time each waiting customer is willing to wait before starting service is exponentially distributed with mean 1 minute, independent of everything else.

(a) (8 points) Construct an appropriate model of this system that can be used to determine the long-run performance.

(b) (7 points) What is the long-run proportion of time that both ATM machines are simultaneously idle?

(c) (7 points) What is the long-run proportion of all potential arrivals (including ones that balk or are blocked) that are served?

(d) (6 points) Using the model constructed in part (a), give an expression (formula) for the joint probability distribution $P(X(4) = 4, X(7) = 2, X(15) = 2)$, where $X(t)$ is the number of people at the ATM, either waiting or being served, at time t and $P(X(0) = 3) = 1$. Give expressions for all quantities used, but it is not necessary to calculate the numerical values.

(e) (6 points) As in the previous part (d), suppose that $P(X(0) = 3) = 1$. What is the expected time until the state of $X(t)$ first changes?

(f) (6 points) As in the previous parts (d) and (e), suppose that $P(X(0) = 3) = 1$. What is the probability that the next two state changes are both arrivals, bringing the total number in the system to 5?

2. Random Art (15 points)

A probabilistic painter decides to paint a large wall by a random process. He puts points on the wall according to a two-dimensional Poisson process (or Poisson random measure) with constant intensity (rate) $\lambda = 2$ points per square foot.

(a) (5 points) What is the probability that there are no points in a particular rectangle on the wall, which is 3 feet wide and 2 feet high?

(b) (5 points) What is the probability that two disjoint 3×2 rectangles of the kind considered in part (a) each have exactly 10 points?

(c) (5 points) Consider an arbitrary fixed position on the wall. What is the probability that the distance from that location to the nearest random point on the wall is greater than 6 inches (1/2 foot)?