IEOR 3106: Introduction to Operations Research: Stochastic Models

SOLUTIONS to Second Midterm Exam

Chapters 5-6 in Ross, Tuesday, November 11, 2008

Open Book: but only the Ross textbook plus the CTMC Notes and one 8×11 page of notes

Justify your answers; show your work.

1. Defects in a Cable (45 points)

Suppose that minor defects are distributed over the length of a cable according to a Poisson process with rate 9 per mile and, independently, major defects are distributed over the same cable according to a Poisson process with rate 1 per mile. Suppose that we start examining a stretch of cable starting from a designated initial point, which we denote by $t = 0$. [Note: Only the last part, part (h), requires a numerical answer.]

(a) (5 points) What is the probability that there are exactly 3 defects, either minor or major, in the interval $[0, 1/2]$?

We need to start by applying a key property of Poisson processes: The sum or superposition of two independent Poisson processes is again a Poisson process, with a rate equal to the sum of the rates. Let $N(t)$ record the total number of defects, either major or minor, in the interval $[0, t]$. Then $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\lambda = 9 + 1 = 10$ per mile. Hence,

$$P(N(t) = k) = \frac{e^{-10t}(10t)^k}{k!} \quad \text{and} \quad P(N(1/2) = 3) = \frac{e^{-5}(5)^3}{3!}$$

(b) (5 points) What is the conditional probability that there are exactly 3 *major* defects in the interval $[0, 1/2]$, given that there are 4 *minor* defects in the interval $[0, 1/2]$?

Since the two original Poisson processes are independent, the conditioning does not alter the probability. But the Poisson process counting major defects has rate 1. Here we have

$$P(N_{major}(1/2) = 3 | N_{minor}(1/2) = 4) = P(N_{major}(1/2) = 3) = \frac{e^{-0.5}(0.5)^3}{3!}$$

(c) (5 points) What is the probability that the second defect (after 0) is a major defect?

It is convenient to think of all defects, either major or minor, as occurring according to a Poisson process with rate 10. Then the type of successive defects become independent and identically distributed Bernoulli random variables, each being major with probability

$$\frac{1}{9+1} = \frac{1}{10}.$$

(d) (5 points) Let T_3 be the location of the third defect (after 0, either minor or major). What is $E[T_3]$?

The times between successive defects, say X_i, are independent and identically distributed exponential random variables, each with mean 1/10 mile. That is, $T_3 = X_1 + X_2 + X_3$, where $E[X_i] = 1/10$. Hence,

$$E[T_3] = \frac{3}{10} = 0.3$$

(e) (5 points) What is the variance of T_3?

Since T_3 is the sum of three i.i.d. exponential random variables, the variance is the sum of the variances, where $Var(X_i) = E[X_i]^2 = 0.01$. Hence,

$$Var(T_3) = 3(0.01) = 0.03$$

(f) (6 points) What is the probability that there are exactly 2 *major* defects in the interval $[0, 3]$, given that there are exactly 6 *major* defects in the interval $[0, 10]$?

Given that there are exactly 6 major defects in the interval $[0, 10]$, the six major defects are distributed as 6 independent random variables, each uniformly distributed over the interval $[0, 10]$. The location of each of the six defects, then, is Bernoulli, falling in the subinterval $[0, 3]$ with probability 0.3. The conditional number falling into the interval $[0, 3]$ then is binomial.

$$P(N_{major}(3) = 2 | N_{major}(10) = 6) = \left(\frac{6!}{2!4!}\right)(0.3)^2(0.7)^4$$

(g) (7 points) What is the probability that there are exactly 2 *major* defects in the interval $[0, 3]$, given that there are exactly 6 defects of any kind, either major or minor, in the interval $[0, 10]$?

There are several ways to do this, some more elegant than others. Let us start with a straightforward approach: First, for the total number of defects, we have

$$P(N(3) = k | N(10) = 6) = \left(\frac{6!}{k!(6-k)!}\right)(0.3)^k(0.7)^{(6-k)}$$

just as for the major defects in the previous part. Then given that the total number of defects in the interval $[0, 3]$ is k, the number of major defects in that interval is binomial:

$$P(N_{major}(3) = 2 | N(3) = k) = \left(\frac{k!}{2!(k-2)!}\right)(0.1)^2(0.9)^{(k-2)}$$

We then combine these two steps to get an overall expression for the probability:

$$\begin{aligned} P(N_{major}(3)=2|N(10)=6) &= \sum_{k=2}^{k=6} P(N(3)=k|N(10)=6)P(N_{major}(3)=2|N(3)=k) \\ &= \sum_{k=2}^{k=6}\left(\left(\frac{6!}{k!(6-k)!}\right)(0.3)^k(0.7)^{(6-k)}\right)\left(\frac{k!}{2!(k-2)!}\right)(0.1)^2(0.9)^{(k-2)} \end{aligned}$$

It would be OK to stop there. It can easily be computed. However, it is possible to simplify this expression. By performing a change of variables, we can write this as a sum over k ranging from 0 to 4. If we factor out a few constants, then we get

$$\begin{aligned} P(N_{major}(3)=2|N(10)=6) &= \sum_{k=2}^{k=6}\left(\left(\frac{6!}{k!(6-k)!}\right)(0.3)^k(0.7)^{(6-k)}\right)\left(\frac{k!}{2!(k-2)!}\right)(0.1)^2(0.9)^{(k-2)} \\ &= \frac{(0.1)^2(0.3)^2(6\times 5)}{2}\sum_{k=0}^{k=4}\left(\left(\frac{4!}{k!(4-k)!}\right)(0.27)^k(0.7)^{(4-k)}\right) \\ &= \left(\frac{6!}{4!2!}\right)(0.97)^4(0.03)^2 \end{aligned}$$

which is a binomial probability $b(2;6,0.03)$. The second line is after the change of variables. The third line is by applying the binomial theorem; it makes the sum equal to $(0.27+0.70)^4$. The final line is just putting everything in a recognizable form.

Here is a better way. It gets there more directly. Do the conditioning in the other order:

$$\begin{aligned} P(N(10)=6|N_{major}(3)=2) &= P\left(N_{major}(10)-N_{major}(3)+N_{minor}(10)=4|N_{major}(3)=2\right) \\ &= P(N_{major}(10)-N_{major}(3)+N_{minor}(10)=4) \end{aligned}$$

by the independence properties. Since $N_{major}(10)-N_{major}(3)$ and N_{minor} are independent Poisson variables with means 7 and 97, respectively, the above probability is the Poisson probability with mean 97:

$$P(N_{major}(10)-N_{major}(3)+N_{minor}=4)=\frac{e^{-97}(97)^4}{4!}$$

Hence, we can write

$$P(N_{major}(3)=2|N(10)=6)=\frac{P(N(10)=6|N_{major}=2)P(N_{major}(3)=2)}{P(N(10)=6}.$$

When we substitute in the three Poisson probabilities, we get the same answer above, namely,

$$P(N_{major}(3)=2|N(10)=6)=\left(\frac{6!}{4!2!}\right)(0.97)^4(0.03)^2.$$

(h) (7 points) What is the approximate probability that there are more than 120 major defects in the interval $[0,100]$? Is that probability greater than 0.05?

The idea here is to use a normal approximation. (Let $N(\mu, \sigma^2)$ denote a normal random variable with mean μ and variance σ^2.) There are two nearly equivalent ways to get it. We can focus on the Poisson counting process or the partial sums of random variables.

First, we consider the Poisson process view: The number of major defects in the interval $[0, 100]$ has a Poisson distribution with mean $100 \times 1 = 100$. Thus it has variance 100 and standard deviation $\sqrt{100} = 10$. We now use the normal approximation for the Poisson distribution, which is justified because the mean is relatively large:

$$\begin{aligned} P(N(100) > 120) &\approx P\left(N(100, 100) > 120\right) \quad \text{by the normal approximation} \\ &= P\left(\frac{N(100, 100) - 100}{10} > \frac{120 - 100}{10}\right) \\ &= P\left(N(0, 1) > 2\right) \approx 0.023 \end{aligned}$$

Hence, $P(N(100) > 120) < 0.05$.

We could also apply the central limit theorem here. Let S_k be the time of the k^{th} major defect. Then S_k is the sum of k i.i.d. exponential random variables, each with mean 1. We could then ask what is the probability that $P(S_{121} \leq 100)$. We have $E[S_{121}] = 121$ and $Var(S_{121}) = 121$, so that the standard deviation is $SD(S_{121}) = \sqrt{121} = 11$. Hence, by the CLT, this sum is approximately normally distributed. We do not get precisely the same numerical value, but it will be close:

$$\begin{aligned} P(S_{121} \leq 100) &\approx P\left(N(121, 121) \leq 100\right) \quad \text{by the normal approximation} \\ &= P\left(\frac{N(121, 121) - 121}{11} \leq \frac{100 - 121}{11}\right) \\ &= P\left(N(0, 1) \leq -(21/11)\right) \approx P\left(N(0, 1) \leq -1.909\right) \approx 0.028 \end{aligned}$$

Hence, $P(S_{121} \leq 100) < 0.05$.

The difference between the two approximations is due to approximation error. These random variables are not exactly normally distributed.

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2. Two ATM Machines in Lerner Hall. (40 points)

There are two Automatic Teller Machines (ATM's) behind the small restaurant on the first floor of Lerner Hall. Suppose that customers arrive to receive service from one of these ATM machines according to a Poisson process with rate 2 per minute. Suppose that half of these arrivals (each with probability 1/2 independent of everything else) will elect to balk (leave immediately instead of joining the queue) if they cannot receive service immediately upon arrival. In addition, suppose that no arrival will stay if there are already 3 people waiting in addition to 2 being served. (We then say that the arrival is blocked.) Suppose that customer service times at these ATM machines are mutually independent exponential random variables with a mean of 1 minute. Suppose that each customer in queue has limited patience, and is willing to stay only a certain random amount of time before starting service, with the length of time depending upon the customer. Suppose that the time each waiting customer is willing to wait before starting service is exponentially distributed with mean 1 minute, independent of everything else.

(a) (8 points) Construct an appropriate model of this system that can be used to determine the long-run performance.

Let $X(t)$ be the number of customers in the system at time t. By above, that can be any integer from 0 to 5, so that there are 6 states. The stochastic process $\{X(t) : t \geq 0\}$ will be a continuous-time Markov chain (CTMC), in fact, even more, it will be a **birth-and-death (BD) process**, as in Section 6.3 of Ross and Section 5 of my CTMC notes. It is good to draw the **rate diagram** for the birth-and-death process $X(t)$ in Figure 1.

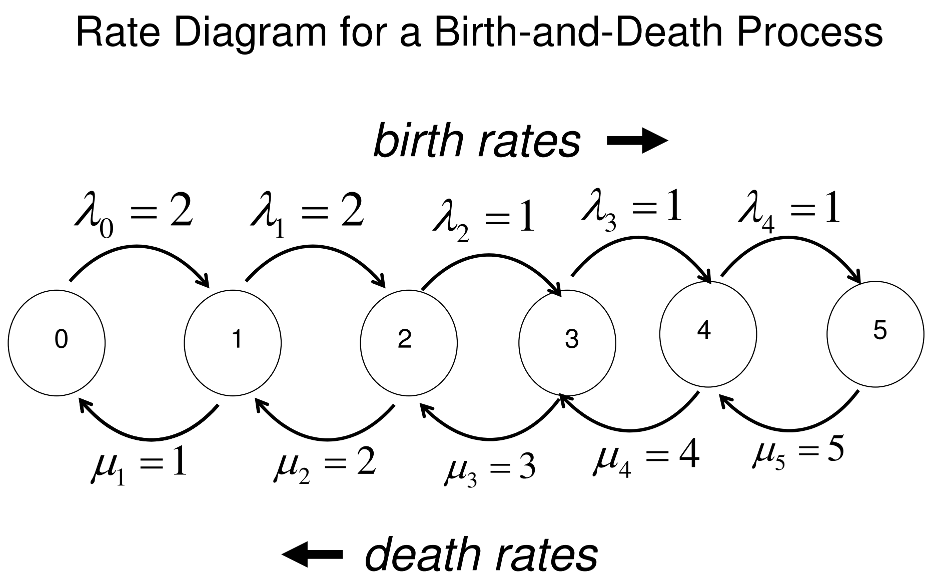

Figure 1: A rate diagram showing the transition rates for the birth-and-death process.

We can also specify the CTMC model by specifying the transition **rate matrix** Q. Here there are 6 states: $0, 1, 2, 3, 4, 5$, representing the number of customers that can be in the system at any time. For this problem, all transitions are either up 1 or down 1, so that this is a BD stochastic process. That means the Q matrix has positive elements going up 1 or down 1 and negative elements on the diagonal. Here the rate matrix, with rates expressed per hour, is:

$$Q = \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ 1 & -3 & 2 & 0 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 & 0 \\ 0 & 0 & 3 & -4 & 1 & 0 \\ 0 & 0 & 0 & 4 & -5 & 1 \\ 0 & 0 & 0 & 0 & 5 & -5 \end{pmatrix}$$

In other words, the birth rates are $\lambda_k \equiv Q_{k,k+1}$, $0 \leq k \leq 4$, and the death rates are $\mu_k \equiv Q_{k,k-1}$, $1 \leq k \leq 5$. The arrival rates for $k \geq 2$ are reduced because of the balking; the arrival rate becomes 1 instead of 2 per minute. The service rate is 1 when only one customer is in the system, but the service rate is 1 per minute for each busy server, so that the total service rate becomes 2 per hours when $k \geq 2$. For $k \geq 3$, there is additional death rate because of the abandonments. Each customer in queue abandons at rate $\theta = 1$ per minute. Hence the death rate in state 4 is $2\mu + 2\theta = 4$ per minute. The diagonal elements of Q are the negative of the off-diagonal row sums.

(b) (7 points) What is the long-run proportion of time that both ATM machines are simultaneously idle?

We need to find the limiting probabilities:

$$\alpha_j \equiv \lim_{t \to \infty} P(X(t) = j).$$

In general for a CTMC, we can find the limiting steady-state probability vector α by solving

$$\alpha Q = 0,$$

which here corresponds to a system of 6 linear equations with 6 unknowns. There is one redundant equation, but we fill that in by requiring that the sum of these probabilities be 1: If we number the states $0, 1, 2, \ldots, 5$, then we need to use the equation $\alpha_0 + \cdots + \alpha_5 = 1.0$. But here we have a BD process, so we can solve the **local-balance (or detailed balance) equations**

$$\alpha_i \lambda_i = \alpha_{i+1} \mu_{i+1}$$

for all i, plus the equation $\alpha_0 + \cdots + \alpha_5 = 1.0$. That leads to the explicit formula

$$\alpha_i = \frac{\frac{\lambda_0 \times \cdots \times \lambda_{i-1}}{\mu_1 \times \cdots \mu_i}}{1 + \sum_{k=1}^{n} \frac{\lambda_0 \times \cdots \times \lambda_{k-1}}{\mu_1 \times \cdots \mu_k}}$$

for $1 \le i \le 5$ and

$$\alpha_0 = \frac{1}{1 + \sum_{k=1}^{n} \frac{\lambda_0 \times \cdots \times \lambda_{k-1}}{\mu_1 \times \cdots \mu_k}} ;$$

see page 371 of Ross. Here we get:

$$\begin{aligned} \alpha_0 &= \frac{30}{176}, \quad \alpha_1 = \frac{60}{176}, \quad \alpha_2 = \frac{60}{176}, \\ \alpha_3 &= \frac{20}{176}, \quad \alpha_4 = \frac{5}{176}, \quad \alpha_5 = \frac{1}{176}. \end{aligned}$$

Finally, we return to our question: The long-run proportion of time that both ATM machines are simultaneously idle is $\alpha_0 = 30/176$.

(c) (7 points) What is the long-run proportion of all potential arrivals (including ones that balk or are blocked) that are served?

That can easily be expressed as the total service rate divided by the total arrival rate. We have

$$\frac{\text{total service rate}}{\text{total arrival rate}} = \frac{1(\alpha_1) + 2(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)}{2} = \frac{116}{176}.$$

(d) (6 points) Using the model constructed in part (a), give an expression (formula) for the joint probability distribution $P(X(4) = 4, X(7) = 2, X(15) = 2)$, where $X(t)$ is the number of people at the ATM, either waiting or being served, at time t and $P(X(0) = 3) = 1$. Give expressions for all quantities used, but it is not necessary to calculate the numerical values.

We apply the Markov property. We use the continuous-time transition matrix $P(t)$, as shown in (2.8) on p. 3 of the CTMC notes.

$$P(X(4) = 4, X(7) = 2, X(15) = 2 | X(0) = 3) = P_{3,4}(4)P_{4,2}(3)P_{2,2}(8),$$

where, for any $t > 0$, the matrix $P(t)$ can be computed by using the matrix expoential function applied to the rate matrix Q given in part (a). In particular, we use the fact that

$$P(t) = e^{Qt}.$$

We need to compute $P(t)$ for three values of t: $t = 3$, 4 and 8. We use the designated elements of $P(t)$ above.

(e) (6 points) As in the previous part (d), suppose that $P(X(0) = 3) = 1$. What is the expected time until the state of $X(t)$ first changes?

The mean holding time in each state i is $1/\nu_i$ where ν_i is the sum of the transition rates out of state i. In particular, $\nu_i = -Q_{i,i}$, so we can read this right off the rate matrix. The expected time of the first state change is thus

$$\frac{1}{\nu_3} = \frac{1}{Q_{3,3}} = \frac{1}{4} \quad \text{minute}$$

(f) (6 points) As in the previous parts (d) and (e), suppose that $P(X(0) = 3) = 1$. What is the probability that the next two state changes are both arrivals, bringing the total number in the system to 5?

The state transitions in a CTMC occur according to the embedded discrete-time Markov chain, as in the first modelling approach, discussed in Section 3.1 of the CTMC notes. If P is the transition matrix of that DTMC, then the answer is

$$P_{3,4}P_{4,5} = \frac{1}{4} \times \frac{1}{5} = \frac{1}{20}.$$

We can reason that the probability of going up from state i is $\lambda_i/(\lambda_i + \mu_i)$. That explains the calculations above. We need to multiply because we need two independent events both to occur.

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2. Random Art (15 points)

A probabilistic painter decides to paint a large wall by a random process. He puts points on the wall according to a two-dimensional Poisson process (or Poisson random measure) with constant intensity (rate) 2 points per square foot.

This problem revisits a homework problem on Homework 6. It is Problem 94 in the textbook, which was assigned. The Poisson random measure was discussed in class on October 16.

(a) (5 points) What is the probability that there are no points in a particular rectangle on the wall, which is 3 feet wide and 2 feet high?

The meaning of the model is that: (i) the number of points in any set has a Poisson distribution with a mean equal to the rate λ multiplied by the area of the set, and (ii) the numbers of points in disjoint sets are independent random variables. Let $N(C)$ be the number of points in this set (rectangle) C. Since the rate has been given to be 2 per square foot and the area of C is $2 \times 3 = 6$ square feet, the mean $[N(C)] = 2 \times 6 = 12$.

$$P(N(C) = 0) = e^{-2\times 6} = e^{-12},$$

not very likely.

(b) (5 points) What is the probability that two disjoint 3×2 rectangles of the kind considered in part (a) each have exactly 10 points?

Because of the independence, we square the probability for each rectangle. (We can use the mean computed from the last part.) The answer is

$$\left(\frac{e^{-12}(12)^{10}}{10!}\right)^2$$

(c) (5 points) Consider an arbitrary fixed position on the wall. What is the probability that the distance from that location to the nearest random point on the wall is greater than 6 inches (1/2 foot)?

As in the homework problem, this is equivalent to there being no points in the set C, where now C is the circle of radius 3. The circle of radius 1/2 has area $\pi r^2 = \pi/4 \approx 0.785$. Hence,

$$P(N(C) = 0) = e^{-\lambda \pi r^2} = e^{-2\times 0.785} = e^{-1.570} \approx 0.208$$
