IEOR 3106: Introduction to Operations Research: Stochastic Models

Part II of First Midterm Exam, Chapter 4, October 12, 2010

There are 3 problems, each with multiple parts.

You need to show your work. Briefly explain your reasoning.

1. A Markov Chain Transition Matrix (15 points)

Consider a Markov chain on the twelve states $\{1, 2, \cdots, 12\}$ with transition matrix P given by

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \begin{pmatrix} 0.4 & 0.1 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.0 & 0.1 & 0.0 & 0.1 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.4 & 0.0 & 0.0 & 0.0 & 0.6 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.7 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.9 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.6 & 0.0 & 0.0 & 0.0 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.1 & 0.0 & 0.0 & 0.1 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 \end{pmatrix}$$

**Note that we are numbering the states 1, 2, ... , 12, with the columns numbered in the same order as the rows.

Please answer the following questions. Two points are subtracted for each wrong answer, up to 15 points.

(a) Which states are accessible from state 1?

(b) From which states is state 1 accessible?

(c) Do states 5 and 6 communicate?

(d) Do states 1 and 6 communicate?

(e) Identify the communication classes for this Markov chain.

(f) Which communication classes are closed? Which are open?

(g) Which states are transient? Which states are recurrent?

(h) Put the transition matrix in canonical form.

In the following questions, we are referring to the states as originally defined and numbered.

(i) Compute the six-step transition probability $P^{(6)}_{6,12}$.

(j) Compute the two-step transition probability $P^{(2)}_{4,10}$.

(k) Compute the two-step transition probability $P^{(2)}_{1,2}$.

(l) Starting in state 3, what is the expected total number of visits to state 7?

(m) Starting in state 1, what is the expected total number of visits to state 5?

2. Random Walk on a Graph (20 points)

Consider the graph shown in Figure 1. There are 7 nodes, labelled with capital letters

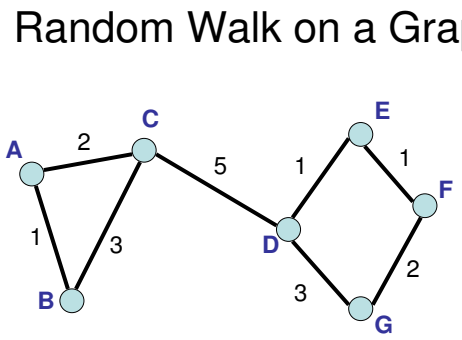

Figure 1: A random walk on a graph.

and 8 arcs connecting some of the nodes. On each arc is a numerical weight. Consider a random walk on this graph, where we move randomly from node to node, always going to a neighbor, via a connecting arc. Let each move be to one of the current node's neighbors, with a probability proportional to the weight on the connecting arc. Thus the probability of going from node C to node A in one step is $2/(2+3+5) = 2/10 = 1/5$, while the probability of moving from node C to node B in one step is $3/10$.

(a) (2 points) What is the probability of going from node A back to to node A in three steps?

(b) (3 points) Starting from node A, what is the probability that the first four steps are to C, then D, then E and finally F?

(c) (2 points) Is the random walk a periodic discrete-time finite-state Markov chain?

(d) (5 points) What is the long-run proportion of moves ending in the node A?

(e) (3 points) Starting from node A, what is the expected number of steps required to return to node A?

For the next parts of the problem, suppose that the random walk starts in node A, but stops the first time it hits either node B or node F. **In the following parts of this question, you are asked to give an expression for the answer; you are not asked to perform the numerical computation**.

(f) (3 points) Give an expression for the expected number of visits to node G before stopping, i.e., before coming to either node B or node F.

(g) (2 points) Give an expression for the probability of eventually stopping in node B.

3. Red and Black: A Game of Chance (15 points)

Simon Hu and Eric Tang visit a casino that has a simplified version of roulette. On each play of the game, a wheel is spun and a ball is dropped and allowed to fall into one of 36 slots. The outcome is one of 36 numbers, of which 18 are red and 18 are black. Simon and Eric decide to study the outcome of red and black on many successive spins to see if there might be some bias in the wheel. Over a very large number of spins, Simon observes that red comes up about 50% of the time. On the other hand, Eric observes that red comes up 60% of the time after red has come up twice in a row. Similarly, Similarly, Eric observes that black comes up 60% of the time after black has come up twice in a row. On the other hand, Eric observes that, after two successive spins yielding different outcomes (colors), red comes up 50% of the time.

(a) (4 points) Make up a Markov chain model based on Eric's observations.

(b) (2 point) With the model in part (a), what is the probability that the outcomes of the next three spins are first red, then black and then red, given that two previous outcomes are both red?

(c) (4 points) Given the model in part (a), calculate the long-run proportion of times that red comes up. Is Eric's observation consistent with Simon's?

(d) (1 point) Suppose that you have the opportunity to wager (bet) on successive spins of the wheel. If you elect to bet, then you pick a color. Suppose that you receive one dollar if you elect to bet on a spin and your color comes up, but you receive nothing if the other color appears. In addition, you must pay 56 cents to bet on that spin. What is your long-run average profit or loss per spin if you play many consecutive games, betting on every spin and betting on red each time?

(e) (2 points) What is your long-run average profit or loss per spin if you play many consecutive games, betting on every spin, making the best possible bet on each spin?

(f) (2 points) Is there a betting strategy where you can make money? If so, what is the maximum long-run average profit per spin over many spins (counting all spins)?

The maximum possible score is 50 points on this part of the first midterm exam.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given not received improper help on this examination," on your examination booklet and sign your name. You may keep the exam itself. Solutions will eventually be posted on line.