IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2010, Professor Whitt, Second Midterm Exam

Chapters 5-6 in Ross, Tuesday, November 16

Open Book: but only the Ross textbook plus the 40-page CTMC Notes

Justify your answers; show your work.

1. Printer Breakdown and Repair (45 points)

Three printer machines – Ricoh, Canon and Office – are maintained by two repairmen. The two repairmen can work on each of the printers if they malfunction, but only one repairman works on any printer at any time. When all three printers have failed, the last one to fail must wait for one of the two repairmen to finish their current job. For simplicity, suppose that the printers and repairmen are potentially working continuously in time, e.g., by counting only the working hours of the business days. Moreover, suppose that the printers have similar properties and the repairmen have similar properties. The time until failure (need for repair), after a printer (any single one of them) has first been installed or has been repaired, is an exponentially distributed random variable with a mean of 2.0 days. The time required to repair a printer after it is failed by any one of the repairmen is an exponentially distributed random variable with a mean of 0.5 days. All the failure times and repair times are mutually independent.

(a) (5 points) Suppose that all three printers are initially working. What is the expected time until the first failure?

(b) (5 points) Suppose that one printer is working and two are under repair. What is the probability that one of the failed printers is repaired before the one working printer fails?

(c) (5 points) Suppose that one printer is working and two are under repair. What is the probability that the both failed printers are repaired before any more failures occur?

(d) (12 points) Determine the long-run proportion of time that k printers are available (working) for each possible k.

(e) (6 points) What is the long-run proportion of time that each printer is available (in working condition)?

(f) (6 points) What is the long-run rate at which failures occur?

(g) (6 points) Suppose that all three printers are initially working. Let $X(t)$ be the number of working printers at time t. Give an expression for the probability $P(X(2) = 2, X(6) = 0, X(13) = 1)$, specifying all quantities used.

2. The Eight (8) Subway Line. (40 points)

A new subway line has been added to the West Side for the convenience of Columbia students. It has six stations. Going north, it starts at 88$^{\text{th}}$ street (station 1) and has stops at 98$^{\text{th}}$ street (station 2), 108$^{\text{th}}$ street (station 3), 118$^{\text{th}}$ street (station 4), 128$^{\text{th}}$ street (station 5) and 138$^{\text{th}}$ street (station 6). It can change tracks and directions at the two end points, so that the trains travel in a loop, going north from station 1 to station 6 and then back south from station 6 to station 1, where it then goes north again. Subway trains follow a strict schedule: The travel time between successive stations is constant, equal to 2 minutes. There are two subway trains, one starting north from station 1 and the other starting south from station 6.

Thus, at station 2, the intervals between successive trains in a specified direction are exactly 10 minutes.

Customers arrive at station i to use the subway according to a Poisson process with rate λ_i per minute. Suppose that the subway has unlimited capacity and that the time to load and unload passengers can be ignored. Suppose that each customer entering station i gets off at station j with probability $P_{i,j}$, independently of all other customers (where $P_{i,i} = 0$). Suppose that people get on subways only in the direction they want to go.

(a) (5 points) Give an expression for the expected number of customers to get on the subway (necessarily going north) at each visit to station 1.

(b) (5 points) Suppose that 8 customers get on the subway at station 1 (necessarily going north) at time t. What is the probability that exactly 3 of these customers had to wait more than 4 minutes before getting on the subway?

(c) (5 points) Give an expression for the long run proportion of customers entering station 3 that get on the north-bound subway (as opposed to the southbound subway).

(d) (5 points) Give an expression for the expected number of customers to get off the northbound subway at each visit to station 4.

(e) (6 points) Give an expression for the probability that the number of customers getting off the northbound subway at a visit to station 4 is exactly j.

(f) (6 points) Give an expression for the probability that, simultaneously, the number of customers getting off the northbound subway at a visit to station 4 is j and the number getting off at the next stop, at station 5, is k.

(g) (8 points) Suppose that $\lambda_i = 2$ for all i and $P_{i,j} = 1/5$ for all j with $j \neq i$. What is the (approximate) probability that the number of customers getting off the northbound subway at one specified visit to station 5 is greater than 20? (Give a numerical answer and explain your reasoning.)

3. Asteroids in Space (15 points)

An astronomer has created a probability model for the distribution of asteroids of at least a given minimal size within the asteroid belt at any given time. The number $N(A)$ of asteroids in a set A in the asteroid belt as a function of the set A has been modeled as a three-dimensional Poisson process (or Poisson random measure) with constant intensity (rate) $\lambda = 3$ asteroids per unit volume (for some appropriate unit of volume, depending on a unit of distance).

(a) (5 points) What is the probability that there are exactly 4 asteroids in the region A with volume $v(A) = 2$ units volume?

(b) (5 points) Suppose that A and B are two disjoint regions in the asteroid belt, with volumes $v(A) = 2$ units volume and $v(B) = 3$ units volume. What is the probability that there are, simultaneously, exactly 4 asteroids in the region A and exactly 5 asteroids in the region B?

(c) (5 points) As in part (b), suppose that A and B are two regions in the asteroid belt, with volumes $v(A) = 2$ units volume and $v(B) = 3$ units volume. However, now suppose that the sets A and B are not disjoint. Suppose instead that the intersection is $A \bigcap B = C$, where the volume of C is $v(C) = 1$ units volume. Now what are the mean and variance of $N(A) + N(B)$?