IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2010, Professor Whitt, Second Midterm Exam

SOLUTIONS

Open Book: but only the Ross textbook plus the 40-page CTMC Notes

Justify your answers; show your work.

1. Printer Breakdown and Repair (45 points)

Three printer machines – Ricoh, Canon and Office – are maintained by two repairmen. The two repairmen can work on each of the printers if they malfunction, but only one repairman works on any printer at any time. When all three printers have failed, the last one to fail must wait for one of the two repairmen to finish their current job. For simplicity, suppose that the printers and repairmen are potentially working continuously in time, e.g., by counting only the working hours of the business days. Moreover, suppose that the printers have similar properties and the repairmen have similar properties. The time until failure (need for repair), after a printer (any single one of them) has first been installed or has been repaired, is an exponentially distributed random variable with a mean of 2.0 days. The time required to repair a printer after it is failed by any one of the repairmen is an exponentially distributed random variable with a mean of 0.5 days. All the failure times and repair times are mutually independent.

Let $X(t)$ be the number of working printers. The random variables $X(t)$ can assume one of the values 0, 1, 2 or 3. Thus there are 4 states. Just like Example 3.2 in the CTMC notes, the system can be analyzed using a CTMC. However, unlike Example 3.2, the stochastic process $\{X(t) : t \geq 0\}$ is itself directly a Markov process. It can be directly represented as a CTMC, in fact as a birth-and-death (BD) process, because the failure and repair rates do not depend on the specific machine or the specific repairman. Instead of the complex rate diagram in Figure 1 in the CTMC notes, we have the simple BD rate diagram in Figure 1 below.

Since the mean repair time is 0.5 days, the rate of repair for each machine, when worked on by a repairman, is 2. In states 0 and 1, when both repairmen are working, the birth rate is $\lambda_0 = \lambda_1 = 2 \times 2 = 4$. In state 2, only one machine is under repair, so the birth rate is $\lambda_2 = 2$. There is no repair going in in state 3 when all machines are working.

Since the mean time for a machiine to fail is 2 days, the fairure rate of each machine is 0.5 per day. In state 3 there are three working machines, each of which can fail. Thus the death rate in state 3 is $\mu_3 = 3 \times 0.5 = 1.5$. In state 2, there are two working machines, so that the death rate is $\mu_2 = 2 \times 0.5 = 1.0$. Overall, the rates are given on the rate diagram.

We can also specify the CTMC model by specifying the transition **rate matrix** Q. Here there are 4 states: $0, 1, 2, 3$, representing the number of working printers at any time. For this problem, all transitions are either up 1 or down 1, so that this is a BD stochastic process. That means the Q matrix has positive elements going up 1 or down 1 and negative elements on the diagonal. That is, the birth rates are $Q_{i,i+1} = \lambda_i$, the death rates are $Q_{i,i-1} = \mu_i$ and the diagonal elements are minus the off-diagonal row sum; i.e. $Q_{i,i} = -(\lambda_i + \mu_i)$. Here the

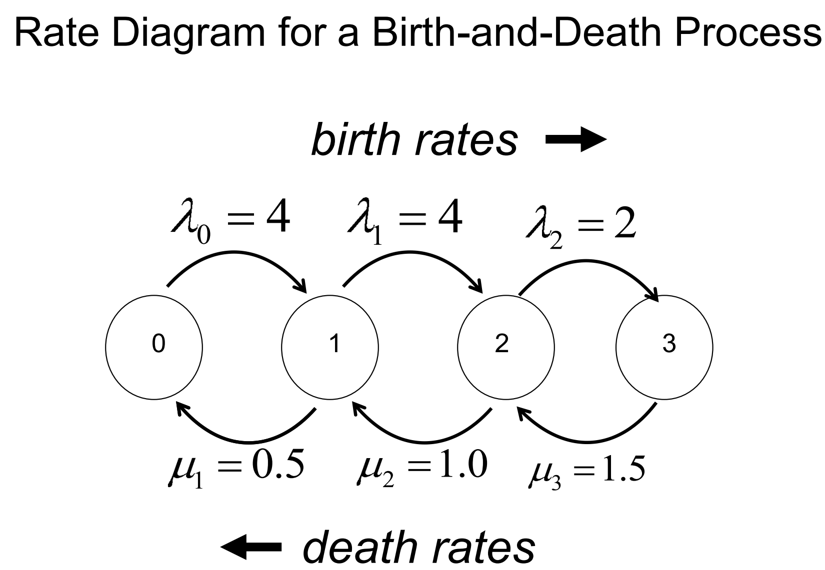

Figure 1: A rate diagram showing the transition rates for the birth-and-death process.

rate matrix, with rates expressed per hour, is:

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} -4 & 4 & 0 & 0 \\ 1/2 & -4.5 & 4 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 3/2 & -3/2 \end{pmatrix}$$

(a) (5 points) Suppose that all three printers are initially working. What is the expected time until the first failure?

The time to failure for each printer is an exponential random variable with a mean of 2 days. The time T to the first failure of any printer is thus the minimum of three independent exponential random variables, each with rate $1/2$ (the reciprocal of the mean). This minimum is also exponential with a rate equal to the sum of the rates. Thus the overall rate is $3/2$. Hence, the mean is its reciprocal

$$E[T] = 2/3 \quad \text{days}.$$

(b) (5 points) Suppose that one printer is working and two are under repair. What is the probability that one of the failed printers is repaired before the one working printer fails?

There are two events that can happen next: (i) the one working printer can fail, (ii) one of the two failed printers can be repaired. These are two independent exponential random variables. The probability of the event that occurs is proportional to its rate. The probability that the next event is a repair is the birth rate λ_1 divided by the sum of the birth rate and the death rate.

$$P(\text{next event is completion of a repair}) = \frac{\lambda_1}{\lambda_1 + \mu_1} = \frac{4}{4 + 1/2} = \frac{8}{9}.$$

(c) (5 points) Suppose that one printer is working and two are under repair. What is the probability that the both failed printers are repaired before any more failures occur?

We must have the event in part (b) followed by another repair. By the lack of memory property, we start over after that first repair. So we have the product of two probabilities.

$$\begin{aligned} P(\text{next event two events are repairs}) &= \frac{\lambda_1}{\lambda_1 + \mu_1} \times \frac{\lambda_2}{\lambda_2 + \mu_2} \\ &= \frac{4}{4 + 1/2} \times \frac{2}{2 + 1} = \frac{16}{27}. \end{aligned}$$

(d) (12 points) Determine the long-run proportion of time that k printers are available (working) for each possible k.

Now we need to solve for the steady state probabilities, $P(X(\infty) = k)$, i.e., the limit of $P(X(t) = k)$ as $t \to \infty$. That is given by α_k, where $\alpha Q = 0$ and $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$. However, by Theorem 5.2 of the CTMC notes, we have

$$\alpha_k = \frac{r_k}{\sum_{i=0}^{i=3} r_i},$$

where $r_0 = 1$, $r_1 = \lambda_0/\mu_1 = 8$, $r_2 = \lambda_0\lambda_1/\mu_1\mu_2 = 32$ and $r_3 = \lambda_0\lambda_1\lambda_2/\mu_1\mu_2\mu_3 = 128/3$. Hence,

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \left(\frac{3}{251}, \frac{24}{251}, \frac{96}{251}, \frac{128}{251}\right).$$

(e) (6 points) What is the long-run proportion of time that each printer is available (in working condition)?

The long-run proportion of time each printer is available (working) is

$$\alpha_3 + \frac{2\alpha_2}{3} + \frac{\alpha_1}{3} = \frac{128 + 64 + 8}{251} = \frac{200}{251}.$$

In state 2, two printers are working. Hence any one printer will be working two thirds of the time that the system is in state 2 in the long run. Similarly, in state 1, one printer is working.

Hence any one printer will be working one third of the time that the system is in state 1 in the long run.

(f) (6 points) What is the long-run rate at which failures occur?

The long-run rate at which failures occur is

$$\alpha_3 \times \mu_3 + \alpha_2 \times \mu_2 + \alpha_1 \times \mu_1 = \frac{128}{251} \times 1.5 + \frac{96}{251} \times 1.0 + \frac{24}{251} \times 0.5 = \frac{300}{251}.$$

(The system is in state j a proportion α_j of the time, and when it is in state j failures occur at rate μ_j.)

(g) (6 points) Suppose that all three printers are initially working. Let $X(t)$ be the number of working printers at time t. Give an expression for the probability $P(X(2) = 2, X(6) = 0, X(13) = 1)$, specifying all quantities used.

This is asking for the joint distribution at specified times. The answer is a special case of (2.8) in the CTMC notes. Here

$$P(X(2) = 2, X(6) = 0, X(13) = 1 | X(0) = 3) = P_{3,2}(2)P_{2,0}(4)P_{0,1}(7),$$

where, for any t, in matrix notation,

$$P(t) = e^{Qt} = \sum_{k=0}^{\infty} \frac{Q^n t^n}{n!},$$

where the rate matrix Q is specified above and $P_{i,j}(t)$ is the element in row i and column j of the matrix $P(t)$. The matrix $P(t)$ could also be calculated by solving the system of differential equations given by $\dot{P}(t) = QP(t) = P(t)Q$.

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2. The Eight (8) Subway Line. (40 points)

A new subway line has been added to the West Side for the convenience of Columbia students. It has six stations. Going north, it starts at 88^{th} street (station 1) and has stops at 98^{th} street (station 2), 108^{th} street (station 3), 118^{th} street (station 4), 128^{th} street (station 5) and 138^{th} street (station 6). It can change tracks and directions at the two end points, so that the trains travel in a loop, going north from station 1 to station 6 and then back south from station 6 to station 1, where it then goes north again. Subway trains follow a strict schedule: The travel time between successive stations is constant, equal to 2 minutes. There are two subway trains, one starting north from station 1 and the other starting south from station 6. Thus, at station 2, the intervals between successive trains in a specified direction are exactly 10 minutes.

Customers arrive at station i to use the subway according to a Poisson process with rate λ_i per minute. Suppose that the subway has unlimited capacity and that the time to load and unload passengers can be ignored. Suppose that each customer entering station i gets off at station j with probability $P_{i,j}$, independently of all other customers (where $P_{i,i} = 0$). Suppose that people get on subways only in the direction they want to go.

(a) (5 points) Give an expression for the expected number of customers to get on the subway (necessarily going north) at each visit to station 1.

Let $N_1(t)$ be the number of customers that arrive to station 1 in the time interval $[0, t]$. This is a Poisson process with rate λ_1. The times between successive subways at station 1 is 10 minutes. Each subway comes from station 2 and then heads back north to station 2. The expected number that get on a subway at each visit to station 1 is the expected number of arrivals over an interval of length 10 minutes. Hence the mean is

$$E[N_1(10)] = 10\lambda_1.$$

(b) (5 points) Suppose that 8 customers get on the subway at station 1 (necessarily going north) at time t. What is the probability that exactly 3 of these customers had to wait more than 4 minutes before getting on the subway?

Given that 8 customers got on the subway at time t, there must have been 8 arrivals in the 10-minute interval $[t - 10, t]$. Given this number, the actual arrival times of the 8 customers are distributed as independent random variables, each uniformly distributed over the ten-minute interval. The probability that each customer had to wait more than 4 minutes is thus $6/10 = 0.6$. The probability that exactly 3 of these customers had to wait more than 4 minutes before getting on the subway is given by the binomial probability

$$b(3; 8, 0.6) = \frac{8!}{3!5!}(0.6)^3(0.4)^5 = 0.12386$$

(This is exploiting one of the basic properties of the Poisson process.)

(c) (5 points) Give an expression for the long run proportion of customers entering station 3 that get on the north-bound subway (as opposed to the southbound subway).

That is the same as the probability of going north for any one customer:

$$P_{3,4} + P_{3,5} + P_{3,6}.$$

A customer going to one of the stations 4, 5 or 6 from station 3 will take the northbound subway. This is using the property that $P_{3,1} + P_{3,2} + P_{3,4} + P_{3,5} + P_{3,6} = 1$.

(d) (5 points) Give an expression for the expected number of customers to get off the northbound subway at each visit to station 4.

Let D_4 be the number of people to get off the northbound subway at station 4. The expected number of customers to is

$$E[D_4] = 10\lambda_1 P_{1,4} + 10\lambda_2 P_{2,4} + 10\lambda_3 P_{3,4}$$

(The expected number of customers that entered station i that get on the northbound subway and get off at station 4 is $10\lambda_i P_{i,4}$ provided that $1 \le i < 4$.

(e) (6 points) Give an expression for the probability that the number of customers getting off the northbound subway at a visit to station 4 is exactly j.

The number of people to get on the northbound subway at station at station i that get off at station 4 is an independent thinning (with probability $P_{i,4}$) of the number that gets on the northbound subway at station i, which has mean $10\lambda_i$, and is thus itself a Poisson random variable with mean $10\lambda_i P_{i,4}$. The numbers for different starting stations i are independent Poisson random variables, because the arrival processes at the different stations are independent Poisson processes. Finally, the sum of independent Poisson random variables is again a Poisson random variable with a mean equal to the sum of the component means. Thus,

$$P(D_4 = j) = \frac{e^{-m_4} m_4^j}{j!}$$

where

$$m_4 = E[D_4] = 10\lambda_1 P_{1,4} + 10\lambda_2 P_{2,4} + 10\lambda_3 P_{3,4},$$

from part (d).

(f) (6 points) Give an expression for the probability that, simultaneously, the number of customers getting off the northbound subway at a visit to station 4 is j and the number getting off at the next stop, at station 5, is k.

Independent thinnings of Poisson random variables become independent Poisson random variables. Thus these are independent random variables. Hence,

$$P(D_4 = j, D_5 = k) = \frac{e^{-m_4} m_4^j}{j!} \times \frac{e^{-m_5} m_5^j}{j!}$$

where m_4 is defined in part (e) and

$$m_5 = E[D_5] = 10\lambda_1 P_{1,5} + 10\lambda_2 P_{2,5} + 10\lambda_3 P_{3,5} + 10\lambda_4 P_{4,5}.$$

(g) (8 points) Suppose that $\lambda_i = 2$ for all i and $P_{i,j} = 1/5$ for all j with $j \neq i$. What is the (approximate) probability that the number of customers getting off the northbound subway at one specified visit to station 5 is greater than 20? (Give a numerical answer and explain your reasoning.)

Applying part (f), we can insert these numbers to get

$$m_5 = E[D_5] = 10\lambda_1 P_{1,5} + 10\lambda_2 P_{2,5} + 10\lambda_3 P_{3,5} + 10\lambda_3 P_{3,5} = 4 \times (10 \times 2 \times \frac{1}{5}) = 16.$$

Hence, D_5 has a Poisson distribution with a mean of $m_5 = E[D_5] = 16$. We can now use a normal approximation for the Poisson distribution, which is appropriate when the mean is not too small.

$$\begin{aligned} P(D_5 > 20) &= P\left(\frac{D_5 - E[D_5]}{\sqrt{Var(D_5)}} > \frac{20 - E[D_5]}{\sqrt{Var(D_5)}}\right) \\ &\approx P\left(N(0,1) > \frac{20 - E[D_5]}{\sqrt{Var(D_5)}}\right) = P\left(N(0,1) > \frac{20-16}{\sqrt{16}}\right) \\ &= P(N(0,1) > 1) \approx 0.16 \end{aligned}$$

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3. Asteroids in Space (15 **points)**

An astronomer has created a probability model for the distribution of asteroids of at least a given minimal size within the asteroid belt at any given time. The number $N(A)$ of asteroids in a set A within the asteroid belt as a function of the set A has been modeled as a three-dimensional Poisson process (or Poisson random measure) with constant intensity (rate) $\lambda = 3$ asteroids per unit volume (for some appropriate unit of volume, depending on a unit of distance).

(a) (5 points) What is the probability that there are exactly 4 asteroids in the region A with volume $v(A) = 2$ units volume?

The random variable $N(A)$ has a Poisson distribution with mean $E[N(A)] = 3v(A) = 6$. Hence,

$$P(N(A) = 4) = \frac{e^{-6}6^4}{4!}$$

(b) (5 points) Suppose that A and B are two disjoint regions in the asteroid belt, with volumes $v(A) = 2$ units volume and $v(B) = 3$ units volume. What is the probability that there are, simultaneously, exactly 4 asteroids in the region A and exactly 5 asteroids in the region B?

For a three-dimensional Poisson process, the number of points in disjoint sets are independent random variables. Hence,

$$P(N(A) = 4, N(B) = 5)) = P(N(A) = 4)P(N(B) = 5)) = \left(\frac{e^{-6}6^4}{4!}\right)\left(\frac{e^{-9}9^5}{5!}\right)$$

(c) (5 points) As in part (b), suppose that A and B are two regions in the asteroid belt, with volumes $v(A) = 2$ units volume and $v(B) = 3$ units volume. However, now suppose that the sets A and B are not disjoint. Suppose instead that the intersection is $A \bigcap B = C$, where the volume of C is $v(C) = 1$ units volume. Now what are the mean and variance of $N(A) + N(B)$?

First, the mean of a sum is the sum of the means, even if the random variables are dependent. Hence,

$$E[N(A) + N(B)] = E[N(A)] + E[N(B)] = 6 + 9 = 15.$$

Second, observe that

$$N(A) = N(A - C) + N(C) \quad \text{and} \quad N(B) = N(B - C) + N(C),$$

where the three sets $A - C \equiv A \bigcap C^c$, C and $B - C \equiv B \bigcap C^c$ are disjoint, with C^c denoting the complement of the set C. Hence, $N(A-C)$, $N(C)$ and $N(B-C)$ are independent Poisson random variables with means 3, 3 and 6, respectively. Hence,

$$\begin{aligned}
Var(N(A) + N(B)) &= Var(N(A - C) + 2N(C)) + N(B - C)) \\
&= Var(N(A - C)) + Var(2N(C))) + Var(N(B - C)) \\
&= Var(N(A - C)) + 4Var(N(C))) + Var(N(B - C)) \\
&= E[N(A - C)] + 4E[N(C)] + E[N(B - C)] = 3 + 12 + 6 = 21.
\end{aligned}$$

In the second line we use the property that the variance of the sum of independent random variables is the sum of the variances. In the third line we use the propoerty that $Var(cX) = c^2 Var(X)$ for a random variable X and a constant c. For the fourth (final) line, we use the fact that the variance equals the mean for Poisson random variables. And we have determined the means.
