

IEOR 3106: Introduction to Operations Research: Stochastic Models
Part I of First Midterm Exam, Chapters 1-3, October 5, 2010

There are 4 problems, each with multiple parts.

You need to show your work. Briefly explain your reasoning.

1. A Game of Chance (15 points)

Consider a game of chance played by making independent flips of a single fair coin. On each flip the coin comes up heads with probability $1/2$. The game is played in stages. There are up to three stages.

In the first stage, the player flips the coin two times and counts the number of heads observed in these two flips. Let N_1 be the random number of heads observed in stage 1. The game is over if $N_1 = 0$.

The game continues to a second stage if $N_1 \geq 1$. In the second stage, the player flips the coin N_1 times and counts the number of heads observed in these N_1 flips. Let N_2 be the random number of heads observed in stage 2. The game is over if $N_2 = 0$.

If $N_2 \geq 1$, then the game continues to a third stage. In the third stage, the player flips the coin N_2 times and counts the number of heads observed in these N_2 flips. Let N_3 be the random number of heads observed in stage 3.

Let N be the number of heads observed in the last stage played. For example, $N = N_1 = 0$ if $N_1 = 0$, while $N = N_3 = 2$ if $N_1 \geq 1$, $N_2 \geq 1$ and $N_3 = 2$.

- (a) (5 points) What is the probability distribution of N ?
- (b) (1 point) What is $E[N]$?
- (c) (2 points) What is the variance of N ?
- (d) (5 points) What is the conditional probability $P(N_1 = 0 | N = 0)$?
- (e) (2 points) What is the joint probability $P(N_1 = 1, N_2 = 1, N_3 = 1)$?

2. The Evolving Price of a Share of Foolsgold, Inc. (10 points)

Consider the following probability model of the evolving price of a share of the stock in a company called Foolsgold, Inc. Let S_n denote the price at the end of day n . At the end of day 0 the stock price is $S_0 = 100$. Suppose that, for each $n \geq 1$, $S_n = S_{n-1} + X_n$, where the daily change X_n is independent of S_{n-1} with $X_n = 0.1 + Y_n$, $n \geq 1$, and the random variables Y_1, Y_2, \dots are independent and identically distributed with $P(Y_n = 1) = 1/12$, $P(Y_n = -2) = 1/24$ and $P(Y_n = 0) = 21/24$.

- (a) (2 points) What is $E[S_{100}]$, the expected stock price at the end of day $n = 100$ days?
- (b) (8 points) What is the approximate probability that the stock price goes up over the first 100 days, i.e., the approximate probability of $P(S_{100} > S_0)$?

3. The Lifetimes of Two Components of a Computer (15 points)

Let X and Y be the random lifetimes of two components of a computer, measures in years. Suppose that the joint probability density function (pdf) of the random vector (X, Y) is

$$f_{X,Y}(x, y) = \frac{4e^{-8x}}{x}, \quad 0 \leq y \leq 2x, \quad x \geq 0.$$

- (a) (4 points) What is the pdf of X ?
- (b) (2 points) What is the moment generating function (mgf) of X ?
- (c) (4 points) What is the conditional pdf of Y given that $X = 3$?
- (d) (2 points) What is $E[Y^2|X = 3]$?
- (e) (3 points) What is the covariance of X and Y ?

4. Playing Until One Player Gets Two Points Ahead (10 points)

Two players, A and B , play a succession of games until one player has won two games more than the other. The outcomes of the games are independent trials.

(a) (4 points) Suppose that the probability A wins each game is p . What is the expected number of games played until one player first succeeds in winning two more games than the other?

(b) (1 point) In part (a) what value of p makes the expected number of games played as large as possible, and how large is that?

(c) (3 points) Suppose that the probability A wins a game changes from p to q whenever player A has won one more game than player B , and changes from p to r whenever player A has won one less game than player B . What is the expected number of games played until one player first succeeds in winning two more games than the other?

(d) (2 points) In part (c), suppose that the probabilities are constrained to assume the following values: $1/5 \leq p \leq 4/5$, $1/4 \leq q \leq 3/4$ and $1/3 \leq r \leq 2/3$. What values of p , q and r make the expected total number of games as large as possible, and what is that maximum expected value?

The maximum possible score is 50 points on this part of the first midterm exam.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given nor received improper help on this examination," on your examination booklet and sign your name. You may keep the exam itself. Solutions will eventually be posted on line.