IEOR 3106: Introduction to Operations Research: Stochastic Models

SOLUTIONS to Part II of First Midterm Exam, October 12, 2010

There are 3 problems, each with multiple parts.

You need to show your work. Briefly explain your reasoning.

1. A Markov Chain Transition Matrix (15 points)

Consider a Markov chain on the twelve states $\{1, 2, \cdots, 12\}$ with transition matrix P given by

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \begin{pmatrix} 0.4 & 0.1 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.0 & 0.1 & 0.0 & 0.1 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.4 & 0.0 & 0.0 & 0.0 & 0.6 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.7 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.9 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.6 & 0.0 & 0.0 & 0.0 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.1 & 0.0 & 0.0 & 0.1 & 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 \end{pmatrix}$$

**Note that we are numbering the states 1, 2, ... , 12, with the columns numbered in the same order as the rows.

Please answer the following questions. Two points are subtracted for each wrong answer, up to 15 points.

(a) Which states are accessible from state 1?

Remember that j being *accessible* from i means that you can get from i to j in some finite number of steps, not necessarily in a single step. It turns out that **all states except state 11** are accessible from state 1. This can be determined by constructing a graph showing the 1-step connectivity. It becomes clear from the canonical form of the transition matrix, in part (h). To answer this part, you should be beginning your construction of that canonical form. You could do part (h) first.

(b) From which states is state 1 accessible?

The principle is the same, but it is easier to leave state 1 than to get to it. In fact, state 1 is accessible only from states 1 and 8.

(c) Do states 5 and 6 communicate?

Yes, states 5 and 6 communicate. First, 5 is accessible from 6 in one step. Second, 6 is accessible from 5 in two steps via 9.

(d) Do states 1 and 6 communicate?

No, 6 is accessible from 1, but 1 is not accessible from 6.

(e) Identify the communication classes for this Markov chain.

Each communication class is a subset of the states. When we form communication classes, we construct a *partition* of the set of states (into disjoint subsets whose union is the whole set). Here there are 7 communication classes: $\{2\}$, $\{3,7\}$, $\{4,10\}$, $\{5,6,9\}$, $\{1,8\}$, $\{11\}$ and $\{12\}$.

(f) Which communication classes are closed? Which are open?

The closed classes are the classes that you cannot leave. The open classes are the classes from which you can leave, and thus eventually will leave with probability 1. There are four closed classes: $\{2\}$, $\{3,7\}$, $\{4,10\}$ and $\{5,6,9\}$. There are three open classes: $\{1,8\}$, $\{11\}$ and $\{12\}$.

(g) Which states are transient? Which states are recurrent?

The states in closed communication classes are recurrent: You will return with probability 1. The states in open communication classes are transient: You will eventually leave for the last time and never return again after that. The transient states are 1, 8, 11 and 12; the others are recurrent.

(h) Put the transition matrix in canonical form.

Here is the canonical form of the transition matrix P:

$$P = \begin{array}{c} 2 \\ 3 \\ 7 \\ 4 \\ 10 \\ 5 \\ 6 \\ 9 \\ 11 \\ 12 \\ 1 \\ 8 \end{array} \left(\begin{array}{cccccccccccc} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.4 & 0.6 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.6 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.7 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.9 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 \\ 0.1 & 0.0 & 0.1 & 0.0 & 0.1 & 0.0 & 0.1 & 0.0 & 0.0 & 0.1 & 0.4 & 0.1 \\ 0.1 & 0.0 & 0.2 & 0.0 & 0.2 & 0.1 & 0.1 & 0.1 & 0.0 & 0.0 & 0.1 & 0.1 \end{array} \right)$$

We have re-ordered the states so that the 4 closed communication classes appear together at the top, while the 3 open communication classes appear together at the bottom. We have grouped the states in the same communication class together. The original states are listed on the left. As before, the columns and rows both have this same ordering. For example, the entry 1.0 in the upper left corner is the transition probability $P_{2,2}$, because state 2 has been moved to being first. We have ordered the four closed communication classes by size, putting the smaller ones first, but that is optional. You could have a different matrix, but the states in the same communication class must appear together, next to each other, and the transient states must appear at the bottom. We ordered the open classes so that the ones closer to absorption appear first. From state 11, the chain is immediately absorbed into 2; from state 12 the chain is immediately absorbed into the set $\{4, 10\}$. From the set $\{1, 8\}$, it is possible to stay there and it is possible to go next to state 12. Hence, the set $\{1, 8\}$ belongs below the set $\{12\}$. But that ordering of the open classes is optional as well.

In the following questions, we are referring to the states as originally defined and numbered.

(i) Compute the six-step transition probability $P^{(6)}_{6,12}$.

$$P^{(6)}_{6,12} = 0$$

(j) Compute the two-step transition probability $P^{(2)}_{4,10}$.

Note that states 4 and 10 belong to the same communicating class, so here there is something to compute. It suffices to look at the little 2×2 sub-matrix, which is a transition matrix in its own right. We have

$$P = \begin{array}{c} 4 \\ 10 \end{array} \left(\begin{array}{cc} 0.7 & 0.3 \\ 0.5 & 0.5 \end{array} \right) .$$

The columns are understood to be labelled in the same way as the rows. To proceed, we calculate the two-step transition matrix $P^{(2)} = P^2$ for this little DTMC. We get

$$P^2 = \begin{array}{c} 4 \\ 10 \end{array} \left(\begin{array}{cc} 0.64 & 0.36 \\ 0.60 & 0.40 \end{array} \right) .$$

Hence, $P^{(2)}_{4,10} = 0.36$.

(k) Compute the two-step transition probability $P^{(2)}_{1,2}$.

Things are a little more complicated here, because 1 is a transient state, while 2 is an absorbing state. But you just need to think clearly: There are *three* possibilities, i.e.,

$$P^{(2)}_{1,2} = P_{1,2}P_{2,2} + P_{1,1}P_{1,2} + P_{1,8}P_{8,2} = (0.1)(1.0) + (0.4)(0.1) + (0.1)(0.1) = 0.15$$

(l) Starting in state 3, what is the expected total number of visits to state 7?

State 3 is a recurrent state, belonging to the same closed communicating class with state 7. Thus the expected total number of visits to state 7, starting in state 3, is infinite.

(m) Starting in state 1, what is the expected total number of visits to state 5?

This might at first look complicated, but you only need remember that infinity times any positive number is infinity, and infinity plus any finite number is infinity. There is a positive probability of going from state 1 to state 6 in one step; once in state 6, the expected number of visits to state 5 is infinite. So the expected number of visits to state 5 starting in state 1 is infinite. This is true even though we cannot get from state 1 to state 5 in one step, and there is a significant chance (probability) that we will *never* reach state 5 from state 1.

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2. Random Walk on a Graph (20 points)

Consider the graph shown in Figure 1. There are 7 nodes, labelled with capital letters and 8 arcs connecting some of the nodes. On each arc is a numerical weight. Consider a random walk on this graph, where we move randomly from node to node, always going to a neighbor, via a connecting arc. Let each move be to one of the current node's neighbors, with a probability proportional to the weight on the connecting arc. Thus the probability of going from node C to node A in one step is $2/(2 + 3 + 5) = 2/10 = 1/5$, while the probability of moving from node C to node B in one step is $3/10$.

(a) (2 points) What is the probability of going from node A back to to node A in three steps?

We can go from A back to A in three steps in exactly two ways: $A \to B \to C \to A$ and $A \to C \to B \to A$. We should add the probabilities of these two events:

$$P(A \to B \to C \to A) = \frac{1}{3} \times \frac{3}{4} \times \frac{2}{10} = \frac{1}{20}$$

while

$$P(A \to C \to B \to A) = \frac{2}{3} \times \frac{3}{10} \times \frac{1}{4} = \frac{1}{20}$$

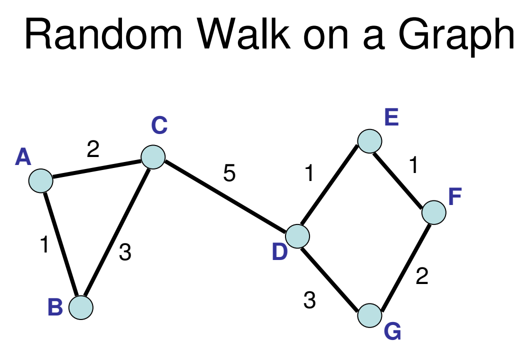

Figure 1: A random walk on a graph.

So the total probability is 1/10.

(b) (3 points) Starting from node A, what is the probability that the first four steps are to C, then D, then E and finally F?

The same reasoning applies here:

$$P(A \to C \to D \to E \to F) = \frac{2}{3} \times \frac{5}{10} \times \frac{1}{9} \times \frac{1}{2} = \frac{1}{54}.$$

(c) (2 points) Is the random walk a periodic discrete-time finite-state Markov chain?

No. The random walk is a discrete-time finite-state Markov chain, but it is not periodic. From node C you can return in 2 moves via D, but also in 3 moves via A and B (in two different ways). There is no $d > 1$ such that the chain can only be in a state at times kd, starting in that state. See Section 4.4 for discussion.

(d) (5 points) What is the long-run proportion of moves ending in the node A?

See Example 4.32 in §4.8. We exploit reversibility. That makes the steady-state probability for each node proportional to the sum of the weights on the arcs out of that node. Thus, $\pi_A = 3/36 = 1/12$.

(e) (3 points) Starting from node A, what is the expected number of steps required to return to node A?

Apply part (d). The expected time to return is the reciprocal of the long-run proportion. See page 225 in the 10^{th} edition. Hence the expected time to return to node A is

$$\frac{1}{\pi_A} = \frac{12}{1} = 12.$$

For the next parts of the problem, suppose that the random walk starts in node A, but stops the first time it hits either node B or node F. **In the following parts of this question, you are asked to give an expression for the answer; you are not asked to perform the numerical computation**.

(f) (3 points) Give an expression for the expected number of visits to node G before stopping, i.e., before coming to either node B or node F.

These last two parts involve an absorbing Markov chain, as in liberating Makov mouse. See Section 4.6. As discussed in the lecture notes, here the absorbing DTMC has the general form:

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} ,$$

where I is an identity matrix (1's on the diagonal and 0's elsewhere) and 0 (zero) is a matrix of zeros. In this case, I would be 2×2, R is 5×2 and Q is 5×5. The matrix Q describes the probabilities of motion among the transient states, as discussed. The matrix R gives the probabilities of absorption in one step (going from one of the transient states to one of the absorbing states in a single step). Here the absorbing states are nodes B and F. In general Q would be square, say m by m, while R would be m by k, and I would be k by k. In particular,

$$Q = \begin{matrix} A \\ C \\ D \\ E \\ G \end{matrix} \begin{pmatrix} 0 & 2/3 & 0 & 0 & 0 \\ 2/10 & 0 & 5/10 & 0 & 0 \\ 0 & 5/9 & 0 & 1/9 & 1/9 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 3/5 & 0 & 0 \end{pmatrix} ,$$

and

$$R = \begin{matrix} A \\ C \\ D \\ E \\ G \end{matrix} \begin{pmatrix} 1/3 & 0 \\ 3/10 & 0 \\ 0 & 1/9 \\ 0 & 1/5 \\ 0 & 2/5 \end{pmatrix} ,$$

The columns of Q are the same as the rows of Q, whereas the columns of R are B and F, in that order.

$$I - Q = \begin{matrix} A \\ C \\ D \\ E \\ G \end{matrix} \begin{pmatrix} 1 & -2/3 & 0 & 0 & 0 \\ -2/10 & 1 & -5/10 & 0 & 0 \\ 0 & -5/9 & 1 & -1/9 & -1/9 \\ 0 & 0 & -1/2 & 1 & 0 \\ 0 & 0 & -3/5 & 0 & 1 \end{pmatrix} .$$

Then the matrix N is the inverse of the matrix $I - Q$ above. The answer then is $N_{A,G}$, where $N = (I - Q)^{-1}$, where A and G are the appropriate states. That is the element in the first row and the last column.

(g) (2 points) Give an expression for the probability of eventually stopping in node B.

Here the answer is $B_{A,B}$, where $B = NR$, and A and B are the state labels. It is the row of N corresponding to the initial state A multiplied by the column of R corresponding to the ending absorbing state B.

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3. Red and Black: A Game of Chance (15 points)

Simon Hu and Eric Tang visit a casino that has a simplified version of roulette. On each play of the game, a wheel is spun and a ball is dropped and allowed to fall into one of 36 slots. The outcome is one of 36 numbers, of which 18 are red and 18 are black. Simon and Eric decide to study the outcome of red and black in many successive spins to see if there might be some bias in the wheel. Over a very large number of spins, Simon observes that red comes up about 50% of the time. On the other hand, Eric observes that red comes up 60% of the time after red has come up twice in a row. Similarly, Eric observes that black comes up 60% of the time after black has come up twice in a row. On the other hand, Eric observes that, after two successive spins yielding different outcomes (colors), red comes up 50% of the time in the next spin.

(a) (4 points) Make up a Markov chain model to describe the probabilities of successive outcomes based on Eric's observations.

This part is like the weather DTMC in Example 4.4, where the probability of the weather on any given day depends on the weather on the previous two days. We let the state be (x_{n-1}, x_n), the outcomes on days $n-1$ and n. The DTMC then transitions from the state (x_{n-1}, x_n) to the state (x_n, x_{n-1}). The model is the transition matrix

$$P = \begin{matrix} RR \\ BR \\ RB \\ BB \end{matrix} \begin{pmatrix} 0.6 & 0 & 0.4 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.4 & 0 & 0.6 \end{pmatrix},$$

where the columns are labeled in the same way, and the same order, as the rows.

(b) (2 points) With the model in part (a), what is the probability that the outcomes of the next three spins are first red, then black and then red, given that two previous outcomes are both red?

This can be represented as the product of three transition probabilities:

$$P_{(R,R),(R,R)} \times P_{(R,R),(R,B)} \times P_{(R,B),(B,R)} = 0.6 \times 0.4 \times 0.5 = 0.12$$

(c) (4 points) Given the model in part (a), calculate the long-run proportion of times that red comes up. Is Eric's observation consistent with Simon's?

We see that this is an irreducible DTMC. It suffices to solve the matrix equation $\pi = \pi P$ with the equation $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$. We get the equations:

$$\begin{aligned} 0.6\pi_1 + 0.5\pi_2 &= \pi_1 \\ 0.5\pi_3 + 0.4\pi_4 &= \pi_2 \\ 0.4\pi_1 + 0.5\pi_2 &= \pi_3 \\ 0.5\pi_3 + 0.6\pi_4 &= \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned}$$

From equation 1, we get $\pi_1 = (5/4)\pi_2$. Then, using this in equation 3, we get $\pi_3 = 0.4(5/4)\pi_2 + 0.5\pi_2 = \pi_2$. From equation 4, we get $\pi_4 = (5/4)\pi_3$. Combining the last two equations, we get $\pi_4 = (5/4)\pi_2$. Hence, all probabilities can be expressed in terms of π_2. The last equation becomes

$$\begin{aligned} \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \\ (5/4)\pi_2 + \pi_2 + \pi_2 + (5/4)\pi_2 &= 1 \\ (18/4)\pi_2 &= 1 \end{aligned}$$

so that $\pi_2 = 4/18 = 2/9$. Hence

$$\pi = (\pi_{RR}, \pi_{BR}, \pi_{RB}, \pi_{BB}) = (5/18, 4/18, 4/18, 5/18).$$

The long-run proportion of time that red comes up is $\pi_{RR} + \pi_{BR} = 5/18 + 4/18 = 9/18 = 1/2$. Hence, Eric's observation is actually consistent with Simon's.

(d) (1 point) Suppose that you have the opportunity to wager (bet) on successive spins of the wheel. If you elect to bet, then you pick a color. Suppose that you receive one dollar if you elect to bet on a spin and your color comes up, but you receive nothing if the other color appears. In addition, you must pay 56 cents to bet on that spin. What is your long-run average profit or loss per spin if you play many consecutive games, betting on every spin and betting on red each time?

Since red comes up 50% of the time, your average reward per spin is 50 cents, but it costs you 56 cents per spin to play. Hence, in the long run, you lose 6 cents per spin. For example, if you bet on $10,000$ consecutive spins in this way, then your expected overall outcome is $-\$600$.

(e) (2 points) What is your long-run average profit or loss per spin if you play many consecutive games, betting on every spin, making the best possible bet on each spin?

Based on part (a), your bet should depend on the previous two outcomes. You bet on R when the state is (R, R); you bet on B when the state is (B, B). Otherwise it does not matter how you bet. Based on part (c), you see that your long run average payoff is

$$0.6\pi_{RR} + 0.5\pi_{BR} + 0.5\pi_{RB} + 0.6\pi_{BB} = 0.6(5/18) + 0.5(4/18) + 0.5(4/18) + 0.6(5/18) = \frac{10}{18} = \frac{5}{9}$$

That is 5/9 dollar or $55.555\ldots$ cents. However, it costs 56 cents per spin to play. So, in the long run, you lose $0.444\ldots$ cents per spin. For example, if you bet on $10,000$ consecutive spins in this way, then your expected overall outcome is $-\$44.44$.

(f) (2 points) Is there a betting strategy where you can make money? If so, what is the maximum long-run average profit per spin over many spins (counting all spins)?

The best strategy is to only bet on the spins when the state is either (R, R) or (B, B). In the long run, on these spins you bet on, you win on average $60 - 56 = 4$ cents per spin. But you are only able to bet on 5/9 of the spins. Over all spins, your long-run average profit per spin is $4 \times (5/9) = 20/9 = 2.222\ldots$ cents per spin. For example, if you bet (and do not bet) on $10,000$ consecutive spins in this way, then your expected overall outcome is $+\$222.22$.

The maximum possible score is 50 points on this part of the first midterm exam.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given not received improper help on this examination," on your examination booklet and sign your name. You may keep the exam itself. Solutions will eventually be posted on line.