IEOR 3106: Introduction to Operations Research: Stochastic Models

SOLUTIONS to Part I of the First Midterm Exam, October 5, 2010

You need to show your work. Briefly explain your reasoning.

1. A Game of Chance (15 points)

Consider a game of chance played by making independent flips of a single fair coin. On each flip the coin comes up heads with probability 1/2. The game is played in stages. There are up to three stages.

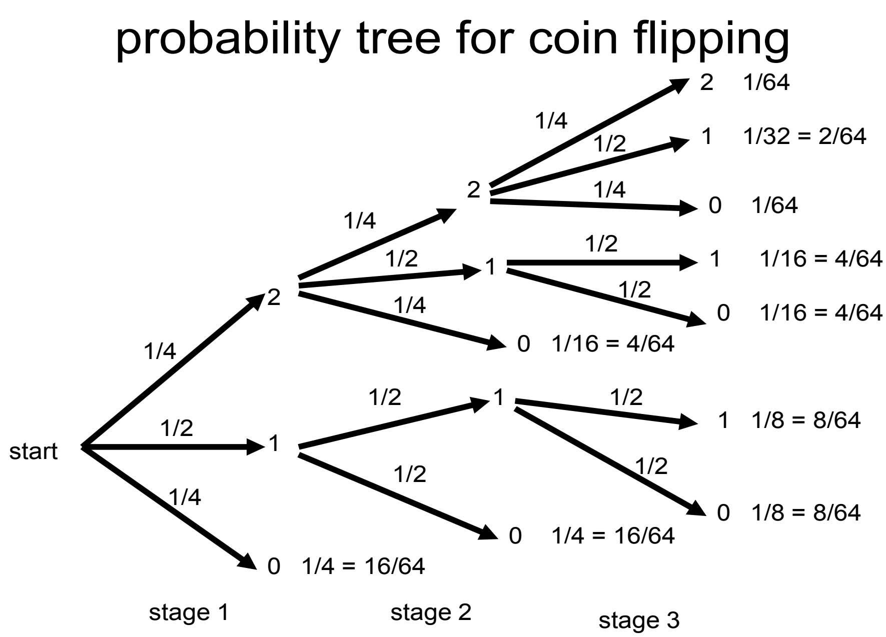
In the first stage, the player flips the coin two times and counts the number of heads observed in these two flips. Let N_1 be the random number of heads observed in stage 1. The game is over if $N_1 = 0$.

The game continues to a second stage if $N_1 \geq 1$. In the second stage, the player flips the coin N_1 times and counts the number of heads observed in these N_1 flips. Let N_2 be the random number of heads observed in stage 2. The game is over if $N_2 = 0$.

If $N_2 \geq 1$, then the game continues to a third stage. In the third stage, the player flips the coin N_2 times and counts the number of heads observed in these N_2 flips. Let N_3 be the random number of heads observed in stage 3.

Let N be the number of heads observed in the last stage played. For example, $N = N_1 = 0$ if $N_1 = 0$, while $N = N_3 = 2$ if $N_1 \geq 1$, $N_2 \geq 1$ and $N_3 = 2$.

It is convenient to draw a probability tree, as in the lecture notes for the first class:



(a) (5 points) What is the probability distribution of N?

From the probability tree, adding the appropriate values on the end of the branches, we get

$$P(N=0)=49/64, \quad P(N=1)=14/64, \quad P(N=2)=1/64.$$

(b) (1 point) What is $E[N]$?

Then

$$E[N]=0\times P(N=0)+1\times P(N=1)+2\times P(N=2)=16/64=1/4.$$

(c) (2 points) What is the variance of N?

It is easy to use

$$Var(N)=E[N^2]-(E[N])^2,$$

where

$$E[N^2]=0^2\times P(N=0)+1^2\times P(N=1)+2^2\times P(N=2)=18/64=9/32.$$

So $Var(N)=9/32-(1/4)^2=7/32$.

(d) (5 points) What is the conditional probability $P(N_1=0|N=0)$?

From the general formula and then the tree,

$$P(N_1=0|N=0)\equiv\frac{P(N_1=0,N=0)}{P(N=0)}=\frac{1/4}{49/64}=\frac{16}{49}.$$

(e) (2 points) What is the joint probability $P(N_1=1,N_2=1,N_3=1)$?

This is a single branch on the tree. In particular, $P(N_1=1,N_2=1,N_3=1)=1/8$.

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2. The Evolving Price of a Share of Foolsgold, Inc. (10 points)

Consider the following probability model of the evolving price of a share of the stock in a company called Foolsgold, Inc. Let S_n denote the price at the end of day n. At the end of day 0 the stock price is $S_0=100$. Suppose that, for each $n\geq 1$, $S_n=S_{n-1}+X_n$, where the daily change X_n is independent of S_{n-1} with $X_n=0.1+Y_n$, $n\geq 1$, and the random variables $Y_1,Y_2,\ldots$ are independent and identically distributed with $P(Y_n=1)=1/12$, $P(Y_n=-2)=1/24$ and $P(Y_n=0)=21/24$.

(a) (2 points) What is $E[S_{100}]$, the expected stock price at the end of day $n=100$ days?

Note that $E[Y_n] = 0$. Hence, $E[X_n] = 0.1$. Thus, $E[S_n] = 100{+}0.1n$. Hence, $E[S_{100}] = 110$.

(b) (8 points) What is the approximate probability that the stock price goes up over the first 100 days, i.e., the approximate probability of $P(S_{100} > S_0)$?

The idea here is to use a normal approximation. This normal approximation is justified by the central limit theorem (CLT), discussed in Section 2.8 and in class. It is important to state these two points to justify your calculations.

By the central limit theorem, S_{100} is approximately normally distributed. $Var(X_n) = Var(Y_n) = E[Y_n^2] = 6/24 = 1/4$. Hence $E[S_{100}] = 110$ and $Var(S_{100}) = 100/4 = 25$. Hence,

$$\begin{aligned}
P(S_{100} > S_0) &= P(S_{100} > 100) = P\left(\frac{S_{100} - E[S_{100}]}{SD(S_{100})} > \frac{100 - E[S_{100}]}{SD(S_{100})}\right) \\
&\approx P\left(N(0,1) > \frac{100 - E[S_{100}]}{SD(S_{100})}\right) \\
&\approx P\left(N(0,1) > \frac{100 - 110}{5}\right) \\
&\approx P\left(N(0,1) > -2\right) = P\left(N(0,1) < 2\right) \approx 0.9772
\end{aligned}$$

from the table on page 82.

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3. The Lifetimes of Two Components of a Computer (15 points)

Let X and Y be the random lifetimes of two components of a computer, measures in years. Suppose that the joint probability density function (pdf) of the random vector (X,Y) is

$$f_{X,Y}(x,y) = \frac{4e^{-8x}}{x}, \quad 0 \le y \le 2x, \quad x \ge 0.$$

(a) (4 points) What is the pdf of X?

$$\begin{aligned}
f_X(x) &= \int f_{X,Y}(x,y)\,dy = \int_0^{2x} \frac{4e^{-8x}}{x}\,dy \\
&= \frac{4e^{-8x}}{x}\int_0^{2x} 1\,dy \\
&= \frac{4e^{-8x}}{x}(2x) \\
&= 8e^{-8x}.
\end{aligned}$$

We recognize that X is an exponential distribution with mean $1/8$. Since time is measured in years, the expected time is 1.5 months.

(b) (2 points) What is the moment generating function (mgf) of X?

$$\begin{aligned}
\psi_X(t) &\equiv E[e^{tX}] = \int_0^\infty e^{tx} f_X(x)\, dx = \frac{8}{8-t} \\
&= \int_0^\infty e^{tx} 8e^{-8x}\, dx = 8\int_0^\infty e^{-(8-t)x}\, dx = \frac{8}{8-t}
\end{aligned}$$

See Example 2.42.

(c) (4 points) What is the conditional pdf of Y given that $X = 3$?

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{4e^{-8x}/x}{8e^{-8x}} = 1/2x, \quad 0 \leq y \leq 2x.$$

That is, given that $X = x$, Y is uniformly distributed on the interval $[0, 2x]$. Hence,

$$f_{Y|X}(y|3) = 1/6, \quad 0 \leq y \leq 6.$$

(d) (2 points) What is $E[Y^2|X = 3]$?

$$E[Y^2|X = 3] = \int_0^6 y^2 (1/6)\, dy = (1/6)(6)^3/3 = 36/3 = 12.$$

(e) (3 points) What is the covariance $Cov(X, Y)$?

Use the expression $Cov(X, Y) = E[XY] - E[X]E[Y]$. Then

$$E[Y] = \int_0^\infty E[Y|X = x] f_X(x)\, dx = \int_0^\infty x f_X(x)\, dx = E[X] = 1/8.$$

Also

$$\begin{aligned}
E[XY] &= \int_0^\infty dx \int_0^{2x} xy f_{X,Y}(x,y)\, dy \\
&= \int_0^\infty 8x^2 e^{-8x}\, dx \\
&= 2/64, \qquad (1)
\end{aligned}$$

because the second line is the second moment of the exponential with mean 1/8. For an exponential random variable, the second moment is twice the square of the mean. Hence,

$$Cov(X, Y) = E[XY] - E[X]E[Y] = (2/64) - (1/64) = (1/64).$$

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4. Playing Until One Player Gets Two Points Ahead (10 points)

Two players, A and B, play a succession of games until one player has won two games more than the other. The outcomes of the games are independent trials.

(a) (4 points) Suppose that the probability A wins each game is p. What is the expected number of games played until one player first succeeds in winning two more games than the other?

This problem can be approached by setting up a single equation in a single unknown, as in the example with the trapped minor in Section 3.4. Let x be the expected time until one of the two players wins two more games than the other. The equation is

$$x = (p^2 + (1-p)^2)2 + 2p(1-p)(2+x) = 2\alpha + (1-\alpha)(2+x).$$

for $\alpha = (p^2 + (1-p)^2)$. Hence, $x = 2/\alpha$; i.e.,

$$x = \frac{2}{p^2 + (1-p)^2}.$$

(b) (1 point) In part a what value of p makes the expected number of games as large as possible, and how large is that?

From part (a), we see that it suffices to make $g(p) \equiv \alpha \equiv \alpha(p) = (p^2+(1-p)^2) = 1-2p+2p^2$ as small as possible. Here we apply calculus and find that the minimum value of α occurs at $p = 1/2$, which is consistent with intuition. In that case, $x = 4$. In more detail,

$$g'(p) \equiv \frac{dg}{dp} = -2 + 4p \quad \text{and} \quad g''(p) = 4.$$

We set $g'(p) = 0$, getting $p = 1/2$. Since $g''(p) = 4 > 0$, we know that the extreme value is a minimum.

(c) (3 points) Suppose that the probability player A wins a game changes from p to q whenever player A has won one more game than player B, and changes from p to r whenever player A has won one less game than player B. What is the expected number of games played until one player first succeeds in winning two more games than the other?

This is a minor variant of the reasoning in part (a). As before, let x be the expected number of games required. Then we have the equation

$$x = 2\alpha + (1-\alpha)(2+x),$$

where $\alpha = pq + (1-p)(1-r)$. Hence,

$$x = \frac{2}{\alpha} = \frac{2}{pq + (1-p)(1-r)}. \qquad (2)$$

(d) (2 points) In part (c), suppose that $1/5 \leq p \leq 4/5$, $1/4 \leq q \leq 3/4$ and $1/3 \leq r \leq 2/3$. What values of p, q and r make the expected total number of games as large as possible, and what is that maximum expected value?

Note that the expression in equation (2) above is decreasing in q and increasing in r. Hence, we can place q at its lower limit and r at its upper limit. That yields

$$x = \frac{2}{(p/4) + (1-p)/3} = \frac{2}{\frac{1}{3} - \frac{p}{12}} \tag{3}$$

as a function of p. From (3), we see that we should make p as large as possible. The largest possible value is 4/5. Using $p = 4/5$, we see that the maximum possible value is

$$x_{max} = \frac{2}{\frac{1}{3} - \frac{4}{60}} = \frac{2}{\frac{4}{15}} = 15/2 = 7.5$$

Note that the maximum value is larger than in part (b). The expected number of games increases when the player who is ahead has a lower chance of winning.

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