IEOR 3106: Introduction to Operations Research: Stochastic Models

Final Exam, Thursday, December 22, 2011

There are 4 problems, each with multiple parts.

You need to show your work. Briefly explain your reasoning.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given not received improper help on this examination," on your examination booklet and sign your name.

You may keep the exam itself. Solutions will eventually be posted on line.

1. Oatpower, Inc. (30 points)

Ever since the 2003 power outage in the northeastern United States, there has been growing investor enthusiasm for the company Oatpower, Inc., which is developing a new way to efficiently generate vast power from ordinary oats. Oatpower claims that it will be possible to generate sufficient power from a single cup of oats to run a subway train for ten years. If Oatpower is successful, subways and elevators will no longer have to depend on America's aging electric power grid. The power generation method is highly secret, but there is a rumor that it is based on a surprising chemical reaction between oats and Raspberry Snapple.

The current price of Oatpower stock is \$100 per share. Suppose that the Oatpower stock price over time (measured in years) can be modelled as the stochastic process $\{S(t) : t \geq 0\}$, where

$$S(t) \equiv 100 + 5B(t), \quad t \geq 0,$$

and $\{B(t) : t \geq 0\}$ is standard (drift zero, unit variance) Brownian motion.

(a) (4 points) Calculate $E[S(4)]$ and $E[S(4)^2]$.

(b) (4 points) Calculate $P(S(4) > 110)$.

(c) (5 points) Let T_s be the first time that the stock price reaches the level s. Calculate $P(T_{110} \leq 4)$.

(d) (5 points) Let $T \equiv \min\{T_{90}, T_{140}\}$. Calculate $E[S(T)]$ and $E[T]$.

(e) (5 points) Calculate $P(T_{90} < T_{140} < T_{80})$.

(f) (3 points) Calculate $E[S(1)|S(4) = 120]$.

(g) (4 points) Calculate $E[S(1)^2|S(4) = 120]$.

2. Random Walk on a Graph (25 points)

Consider the graph shown in Figure 1 on top of the next page. There are 7 nodes, labelled with capital letters and 8 arcs connecting some of the nodes. On each arc is a numerical weight. Consider a random walk on this graph, where we move randomly from node to node, always going to a neighbor, via a connecting arc. Let each move be to one of the current node's neighbors, with a probability proportional to the weight on the connecting arc, independent of the history prior to reaching the current node. Thus the probability of going from node C to node A in one step is $1/(1+3+5) = 1/9 = 1/9$, while the probability of moving from node

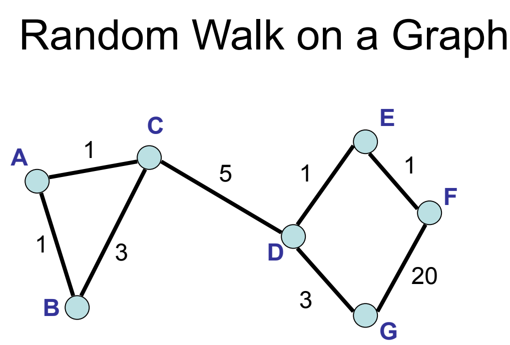

Figure 1: A random walk on a graph.

C to node B in one step is $3/9 = 1/3$. Let X_n be the node occupied after the n^{th} step of the random walk.

(a) (3 points) What is the probability of going from node A back to to node A in two steps?

(b) (3 points) What is the probability of going from node A back to to node A in three steps?

(c) (2 points) Is the stochastic process $\{X_n : n \geq 0\}$ a periodic irreducible discrete-time Markov chain? Why or why not?

(d) (5 points) What is the long-run proportion of moves ending in the node A?

(e) (4 points) Starting from node A, what is the expected number of steps required to return to node A?

(f) (4 points) Give an expression (not the numerical value) for the expected number of visits to node G, starting from node A, before coming to either node B or node F.

(g) (4 points) Give an expression (not the numerical value) for the probability of reaching node B before node F, starting from node A.

3. Back and Forth to Campus (20 points)

Professor Prhab Hubilliti lives at the bottom of the hill on the corner of 117th Street and 7th Avenue. Going each way - up hill to to teach his class at Columbia or down hill back home - Prhab either runs or walks. Going up the hill, Prhab either walks at 2 miles per hour or runs at 4 miles per hour. Going down the hill, Prhab either walks at 3 miles per hour or runs at 6 miles per hour. In each direction, he always runs the entire way or walks the entire way. Since Prhab often works late into the night, he often gets up late, and has to run up hill to get to his class. On any given day, Prhab runs up hill with probability 3/4 and walks up hill with probability 1/4. On the other hand, Prab is less likely to run going back home. On any given day, he runs down hill with probability 1/3 and walks down hill with probability 2/3. The distance in each direction is 1 mile.

(a) (3 points) What is the average speed Prhab goes up the hill to campus when he is going up hill?

(b) (5 points) What is the average time required for Prhab to go up the hill to campus on each trip?

(c) (7 points) What is the long-run proportion of Prhab's total travel time going to and from campus that he spends going up hill to campus?

(d) (5 points) What is the long-run proportion of Prhab's total travel time going to and from campus that he spends walking up hill to campus?

4. The IEOR Printers (25 points)

The IEOR Department has three printers that are maintained by two repairmen. Each printer is working for an exponential length of time with mean 1 week. One repairman works on each failed printer until it is repaired, but the last printer to fail must wait for a repairman to become free whenever all three machines are not working. The repair times are exponential random variables with a mean of 1/2 week. All the failure and repair times are mutually independent. Suppose that all three printers are initially working.

(a) (2 points) If all printers are initially working, then what is the expected time until the first failure?

(b) (5 points) Let $X(t)$ be the number of working printers at time t. Characterize the stochastic process $\{X(t) : t \geq 0\}$, based on the assumptions above.

(c) (3 points) If only one printer is initially working, then what is the probability that one of the other printers is repaired before this working printer fails?

(d) (5 points) What is the long run proportion of time that no printer is working.

(e) (3 points) What is the expected proportion of time each repairman is working on one of the printers?

(f) (3 points) Let $N(t)$ count the number of instants in the interval $[0, t]$ that a printer fails when all three were working. What kind of stochastic process is $\{N(t) : t \geq 0\}$?

(g) (4 points) Let T be the random time between successive instants that a printer fails when all three were working. What is the expected value $E[T]$?