IEOR 3106: Introduction to Operations Research: Stochastic Models

Final Exam, Sunday, December 16, 2012: Three Hours

Four problems, each with multiple parts. Maximum score 100 **(+13 bonus)** $= 113$**.**

You need to show your work. Briefly explain your reasoning. Complicated arithmetic (multiplication and addition) is not required.

Please turn in the exam. Solutions will eventually be posted on line.

1. A Car-Buying Model (25 points)

Mr. Brown has a policy that he buys a new car as soon as his old one breaks down or reaches the age of 6 years, whichever occurs first. Suppose that the successive lifetimes (time until they breakdown) of the cars he buys can be regarded as independent and identically distributed random variables, each uniformly distributed on the interval $[0, 10]$ years. Suppose that each new car costs $\$20,000$. Suppose that Mr. Brown incurs an additional random cost each time the car breaks down. Suppose that this additional breakdown cost is exponentially distributed with mean $\$4,000$. Suppose that he can trade his car in after it is 6 years old if it does not break down, and only if it does not break down, and receive a random dollar value uniformly distributed in the interval $[1000, 3000]$.

(a) (2 points) In the long run, what proportion of the cars Mr. Brown buys break down before they are replaced?

(b) (3 points) What is the mean of the length of time Mr. Brown has each car?

(c) (4 points) What is the variance of the length of time Mr. Brown has each car?

(d) (10 points) What is the long-run average cost per year of Mr. Brown's car-buying strategy?

(e) (6 points) What is the long-run average age of the car currently is use?

2. The IEOR Printers (25 points) $+5$ **bonus**

The IEOR Department has two printers that are maintained by a single repairmen. Assume that time runs continuously, which can be achieved by considering only working hours of the day and week. Each printer is working for an exponential length of time with mean 1 week. The repairman works on each failed printer until it is repaired, so that a second failed printer must wait for the repairman to become free before it can receive attention. Suppose that the expected repair times depends on the printer. The mean time to repair printer 1 is 1 week, while the mean time to repair printer 2 is $1/2$ week. The repair times (assuming the repairman is working constantly) also are exponential. All the failure and repair times are mutually independent. Let $X(t)$ be the number of printers *not* working at time t. Suppose that the two printers are initially working, so that $X(0) = 0$.

(a) (3 points) Let T be the time until the first failure. What is $P(T > 2\,\text{weeks})$?

(b) (5 points) True or false: Indicate whether each of the following five statements is true or false, and briefly explain:

(i) The stochastic process $\{X(t) : t \geq 0\}$ is an irreducible continuous-time Markov chain (CTMC).

(ii) The stochastic process $\{X(t) : t \geq 0\}$ is a birth-and-death process.

(iii) The stochastic process $\{X(t) : t \geq 0\}$ is a reversible CTMC.

(iv) The stochastic process $\{X(t) : t \geq 0\}$ is a renewal process .

(v) The limit $\lim_{t\to\infty} P(X(t) = 2)$ is well defined and can be computed.

(c) (5 points) Carefully define a stochastic model enabling you to compute the long run proportion of time that printer 1 is working.

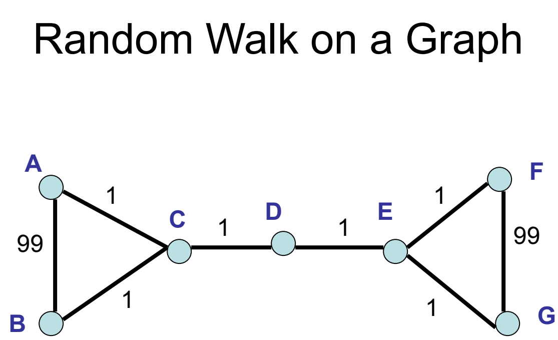
(d) (4 points) Give a mathematical expression for the long run proportion of time that printer 1 is working.

(e) (4 points) Compute the numerical value for the long run proportion of time that printer 1 is working.

(f) (4 points) Let $N(t)$ count the number of instants in the interval $[0, t]$ that a repair is completed, leaving both printers working. What kind of stochastic process is $\{N(t) : t \geq 0\}$? Briefly explain.

(g) (BONUS [harder] 5 points) Let T be the random time between successive instants that a repair is completed, leaving both printers working. What is the expected value $E[T]$?

3. Random Walk on a Graph (25 points) $+4$ **bonus**



Consider the graph shown in the figure above. There are 7 nodes, labeled with capital letters and 8 arcs connecting some of the nodes. On each arc is a numerical weight. Six of the arcs have weight 1, while two of the arcs have weight 99.

Consider a random walk on this graph, where we move randomly from node to node, always going to a neighbor, via a connecting arc. Let each move be to one of the current node's neighbors, with a probability proportional to the weight on the connecting arc, independent of the history prior to reaching the current node. Thus the probability of moving from node A to node C in one step is $1/(1 + 99) = 1/100$, while the probability of moving from node C

to node A in one step is $1/(1+1+1) = 1/3$. Let X_n be the node occupied after the n^{th} step of the random walk. Suppose that $X_0 = A$.

(a) (3 points) What is the probability of going from node A back to to node A in two steps?

(b) (3 points) What is the probability of going from node A back to to node A in three steps?

(c) (7 points) Let π_A be the long-run proportion of moves ending in the node A, and similarly for the other nodes. What is π_A?

(d) (4 points) Starting from node A, what is the expected number of steps required to return to node A?

(e) (4 points) Let $T_{A,D}$ be the first passage time from node A to node D, and similarly for other nodes. Is $E[T_{A,D}] > E[T_{F,D}]$? Justify your answer.

(f) (4 points) Give an expression (not the numerical value) for the expected number of visits to node B, starting from node A, before visiting node F.

(g) (BONUS 4 points) Compute the numerical value of the expected number of visits to node B, starting from node A, before visiting node C. (Hint: set up a simple recursion.)

4. Ten Independent Stocks: Two Investment Strategies (25 points) +4 bonus

You have decided to invest \$800 by buying ten shares at time 0 of stocks initially priced at \$80 per share. Assume that the 10 different stock prices evolve independently over time, with the price of stock j evolving according to the model

$$S_j(t) \equiv 80 + 2.5t + 5B_j(t), \quad t \geq 0,$$

where $\{B_j(t) : t \geq 0\}$ is a standard (drift zero, unit variance) Brownian motion (BM) for each j, with the ten different BM's being stochastically independent.

(a) (2 points) Suppose that you employ a **focused investment strategy** and buy 10 shares of stock 1 at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[10S_1(4)]$ and $Var(10S_1(4))$?

(b) (2 points) With the focused investment strategy in part (a), what is the probability that you will have made a profit? That is, what is the probability that $P(10S_1(4) > 800)$?

(c) (2 points) Suppose that, instead, you decide to employ a **diversified investment strategy** and buy 1 share of each of the 10 different stocks at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[S_1(4) + \cdots + S_{10}(4)]$ and $Var(S_1(4) + \cdots + S_{10}(4))$?

(d) (2 points) With the alternative investment scheme in part (c), what is the probability that you will have made a profit? That is, what is the probability that $P(S_1(4) + \cdots + S_{10}(4) > 800)$?

(e) (5 points) True or false: Indicate whether each of the following five statements is true or false, explaining briefly:

(i) For each of the two investment strategies, the total value of the stock at time $t = 4$ is a random variable with a normal probability distribution.

(ii) The total value of the stock at time $t = 4$ has a probability distribution that is the same for both investment strategies.

(iii) An investor whose sole goal is to maximize his expected return should strongly prefer the diversified investment strategy.

(iv) An investor who wants to achieve the expected return of the focused strategy but minimize his risk, as defined by the probability of suffering a loss over the investment period $[0, 4]$, should strongly prefer the diversified investment strategy.

(v) An investor who wants to maximize the probability that he achieves at least 20% more than the expected value should strongly prefer the focused investment strategy.

(f) (4 points) Let T be the first time that the share price of stock 1 either exceeds its expected value by \$20 or falls below its expected value by \$10; i.e., let

$$T \equiv \inf \{t > 0 : S_1(t) - E[S_1(t)] \geq 20 \quad \text{or} \quad S_1(t) - E[S_1(t)] \leq -10\}.$$

What are $E[S_1(T)]$, $P(S_1(T) - E[S_1(T)] = 20)$ and $E[T]$? Briefly explain.

Problem 4 continued: dependence (in one stock over time and between stocks)

(g) (4 points) What are $E[S_1(3)|S_1(4) = 120]$ and $Var[S_1(3)|S_1(4) = 120]$?

(h) (4 points) What are the expected values: $E[S_1(3)S_1(4)]$ and $E[S_1(4)S_2(4)]$? (The stock price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ are still assumed to be independent in this part.)

(i) (BONUS [harder] 4 points) Suppose that there are stocks for which the stock prices are in fact **stochastically dependent** in various ways. If you could **control the dependence** between the stocks without altering the probability law of each individual stock price process (e.g., by picking the stocks in some clever way), then how could you do better than the diversified investment strategy above? That is, how could you (i) control the dependence and (ii) make an investment strategy in order to reduce the risk (the probability of not making a profit) while leaving the overall expected value of the investment at time 4 (or any other time) unchanged?