IEOR 3106: Introduction to Operations Research: Stochastic Models

SOLUTIONS to Final Exam, Sunday, December 16, 2012

Four problems, each with multiple parts. Maximum score 100 (+13 **bonus**) = 113**.**

You need to show your work. Briefly explain your reasoning. Complicated arithmetic (multiplication and addition) is not required.

Please turn in the exam. Solutions will eventually be posted on line.

1. A Car-Buying Model (25 points)

Mr. Brown has a policy that he buys a new car as soon as his old one breaks down or reaches the age of 6 years, whichever occurs first. Suppose that the successive lifetimes (time until they breakdown) of the cars he buys can be regarded as independent and identically distributed random variables, each uniformly distributed on the interval $[0, 10]$ years. Suppose that each new car costs $\$20,000$. Suppose that Mr. Brown incurs an additional random cost each time the car breaks down. Suppose that this additional breakdown cost is exponentially distributed with mean $\$4,000$. Suppose that he can trade his car in after it is 6 years old if it does not break down, and only if it does not break down, and receive a random dollar value uniformly distributed in the interval $[1000, 3000]$.

(a) (2 points) In the long run, what proportion of the cars Mr. Brown buys break down before they are replaced?

First, note that this problem is a minor variant of **Example 7.12 on p. 440 of the textbook**. Let L be the lifetime of the car. The answer is

$$P(L < 6) = \frac{6}{10}.$$

(b) (3 points) What is the mean of the length of time Mr. Brown has each car?

Let T be the time Mr. Brown has each car. Calculating the mean $E[T]$ is somewhat complicated because the distribution has a (uniform) density over the interval $[0, 6]$ plus a discrete mass at the point 6. However, you can integrate directly with respect to eh uniform density. Using the two-part distribution, we write

$$E[T] = \int_0^6 \frac{x dx}{10} + 6P(L > 6) = \frac{36}{20} + 6(0.4) = 1.8 + 2.4 = 4.2 \, \text{years}$$

If we instead reason more slowly, then we have

$$E[T] = \int_0^6 \frac{x dx}{10} + 6 \int_6^{10} \frac{dx}{10} = \frac{36}{20} + 6(0.4) = 1.8 + 2.4 = 4.2 \, \text{years}$$

(c) (4 points) What is the variance of the length of time Mr. Brown has each car?

First, use the same reasoning is in part (b) above to compute the second moment Then

$$E[T^2] = \int_0^6 \frac{x^2 dx}{10} + 6^2 P(L > 6) = \frac{216}{30} + 36(0.4) = 7.2 + 14.4 = 21.6$$

Then

$$Var(T) = E[T^2] - (E[T])^2 = 21.6 - (4.2)^2 = 21.6 - 17.64 = 4.96$$

(The final arithmetic calculation is not required.)

(d) (10 points) What is the long-run average cost per year of Mr. Brown's car-buying strategy?

We apply renewal reward theory:

$$\text{long run average cost} \quad = \frac{\text{average cost per cycle}}{\text{average length of cycle}}$$

This formula is given in §12 of the formula sheet for renewal theory. We only need the mean turn-in value, which is of course \$2000. Let $\bar{C}$ be the long-run average cost. (The cost can be treated just like the reward. Directly, we can regard the cost as negative reward.) Then

$$\bar{C} = \frac{20,000 + (0.6)4,000 - (0.4)2,000}{E[T]} = \frac{20,000 + 2,400 - 800}{4.2} = \frac{21,600}{4.2} \approx 5143$$

The average cost per year is about \$5143. (Again, the final arithmetic calculation is not required.)

(e) (6 points) What is the long-run average age of the car currently is use?

Let $A(t)$ be the age of the car currently in use at time t. In the long-run, the age of the car currently in use at time t, $A(t)$, is distributed according to the stationary-excess cdf F_e associated with the cdf F of T. The mean of this equilibrium age has the formula

$$\lim_{t \to \infty} E[A(t)] = \frac{E[T^2]}{2E[T]} = \frac{21.6}{2(4.2)} = \frac{21.6}{8.4} = 2.57 \quad \text{years.}$$

This formula is given in §10 of the formula sheet for renewal theory.

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2. The IEOR Printers (25 points)

The IEOR Department has two printers that are maintained by a single repairmen. Assume that time runs continuously, which can be achieved by considering only working hours of the day and week. Each printer is working for an exponential length of time with mean 1 week. The repairman works on each failed printer until it is repaired, so that a second failed printer must wait for the repairman to become free before it can receive attention. Suppose that the expected repair times depends on the printer. The mean time to repair printer 1 is 1 week, while the mean time to repair printer 2 is 1/2 week. The repair times (assuming the repairman is working constantly) also are exponential. All the failure and repair times are mutually independent. Let $X(t)$ be the number of printers *not* working at time t. Suppose that the two printers are initially working, so that $X(0) = 0$.

(a) (3 points) Let T be the time until the first failure. What is $P(T > 2\,\text{weeks})$?

First, notice that this problem is a minor variant of the copier breakdown and repair problem **discussed in class on October 16**. It is taken from **Example 3.2 in the lecture notes on CTMC's**. Of course, this first part is just warmup. Since there are two printers subject to failure, each occurring at rate 1, the total failure rate is 2. Thus,

$$P(T > t) = e^{-2t} \quad \text{and} \quad P(T > 2) = e^{-4}$$

(b) (5 points) True or false: Indicate whether each of the following five statements is true or false, and briefly explain:

(i) The stochastic process $\{X(t) : t \geq 0\}$ is an irreducible continuous-time Markov chain (CTMC).

(ii) The stochastic process $\{X(t) : t \geq 0\}$ is a birth-and-death process.

(iii) The stochastic process $\{X(t) : t \geq 0\}$ is a reversible CTMC.

(iv) The stochastic process $\{X(t) : t \geq 0\}$ is a renewal process .

(v) The limit $\lim_{t\to\infty} P(X(t) = 2)$ is well defined and can be computed.

(i) FALSE, just like Example 3.2 in the CTMC lecture notes, the stochastic process $\{X(t) : t \geq 0\}$ fails to have the Markov property, because the repair times depend on the printer being repaired. In any state except $X(t) = 0$, the future depends on more than $X(t)$. It also depends on which printer is currently under repair.

(ii) FALSE, for the same reason above.

(iii) FALSE, because it fails to be a CTMC.

(iv) FALSE. A renewal process is a counting process, having nondecreasing sample paths.

(v) TRUE, The limit $\lim_{t\to\infty} P(X(t) = 2)$ is well defined and can be computed. That is because we can create another process that is a CTMC and find its limiting distribution. Then we can apply that result to justify the limit of this probability as $t \to \infty$ and calculate its value. Indeed, that is what is done in the next parts.

(c) (5 points) Carefully define a stochastic model enabling you to compute the long run proportion of time that printer 1 is working.

Create a CTMC with states 0, 1, 2, $(1, 2)$ and $(2, 1)$, just as was done for Example 3.2 in the CTMC notes. **Those are appropriate states.** These specify the printers that are *not* working. The state $(1,2)$ means that both printers are not working, but the printer 1 failed first, and so is under repair.

Just as there, we have the rate diagram

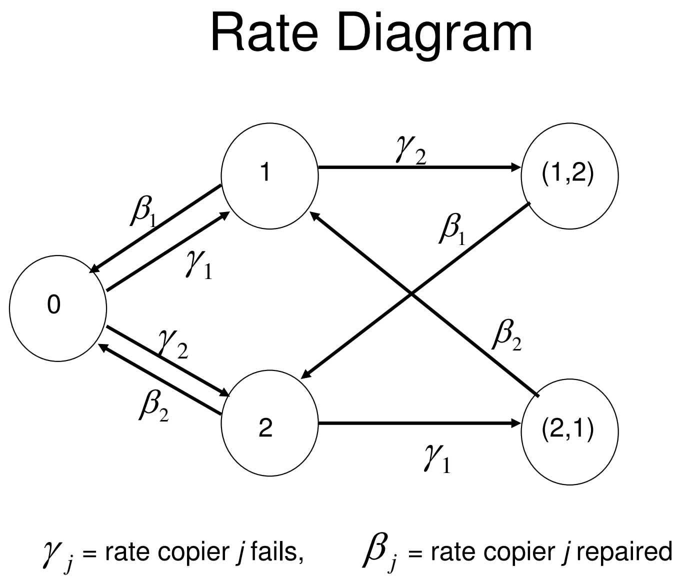

Figure 1: A rate diagram showing the transition rates among the 5 states in for the new CTMC involving printer breakdown and repair.

In other words, the rate matrix (with the columns labeled the same way as the rows) should be

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \\ (1,2) \\ (2,1) \end{array} \left(\begin{array}{ccccc} -(\gamma_1+\gamma_2) & \gamma_1 & \gamma_2 & 0 & 0 \\ \beta_1 & -(\gamma_2+\beta_1) & 0 & \gamma_2 & 0 \\ \beta_2 & 0 & -(\gamma_1+\beta_2) & 0 & \gamma_1 \\ 0 & 0 & \beta_1 & -\beta_1 & 0 \\ 0 & \beta_2 & 0 & 0 & -\beta_2 \end{array} \right) .$$

However, here we have explicit numerical values. In particular, we see that there are 8 possible transitions. The 8 possible transitions should clearly have transition rates

$$\begin{aligned} Q_{0,1} &= \gamma_1 = 1,\ Q_{0,2} = \gamma_2 = 1,\ Q_{1,0} = \beta_1 = 1,\ Q_{1,(1,2)} = \gamma_2 = 1, \\ Q_{2,0} &= \beta_2 = 2,\ Q_{2,(2,1)} = \gamma_1 = 1,\ Q_{(1,2),2} = \beta_1 = 1,\ Q_{(2,1),1} = \beta_2 = 2\ . \end{aligned} \tag{1}$$

Substituting the numerical values, we have the following rate matrix. **Given the states, this is the CTMC model.**

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \\ (1,2) \\ (2,1) \end{array} \left(\begin{array}{ccccc} -2 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 2 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 & -2 \end{array} \right) .$$

(d) (4 points) Give a mathematical expression for the long run proportion of time that printer 1 is working.

Let α be the steady state probability vector for this CTMC defined in part (c). It is the unique probability vector satisfying the matrix equation

$$\alpha Q = 0.$$

That is the steady-state probability vector for the CTMC.

The answer to the question posed is: The long-run proportion of time that printer 1 is working is

$$\alpha_0 + \alpha_2,$$

which is the steady-state probability that no printers are failed plus the steady-state probability that only printer 2 is not working. **That is the full answer, given the CTMC model above.**

(e) (4 points) Compute the numerical value for the long run proportion of time that printer 1 is working.

It only remains to do the numerical computation. We now continue to compute the probability vector α. We write down the equations

$$\begin{aligned} 0 &= -2\alpha_0 + \alpha_1 + 2\alpha_2 \\ 0 &= \alpha_0 - 2\alpha_1 + 2\alpha_{2,1} \\ 0 &= \alpha_0 - 3\alpha_2 + \alpha_{1,2} \\ 0 &= \alpha_1 - \alpha_{1,2} \\ 0 &= \alpha_2 - 2\alpha_{2,1} \end{aligned} \tag{2}$$

From the last two rows, we immediately get

$$\alpha_{1,2} = \alpha_1 \quad \text{and} \quad \alpha_{2,1} = \frac{\alpha_2}{2}.$$

Substituting for $\alpha_{1,2}$ and $\alpha_{2,1}$, we get three equations

$$\begin{aligned} 0 &= -2\alpha_0 + \alpha_1 + 2\alpha_2 \\ 0 &= \alpha_0 - 2\alpha_1 + \alpha_2 \\ 0 &= \alpha_0 + \alpha_1 - 3\alpha_2 \end{aligned} \tag{3}$$

If we subtract the last equation from the one before, then we get

$$-3\alpha_1 + 4\alpha_2 = 0 \quad \text{or} \quad \alpha_2 = (3/4)\alpha_1.$$

If we substitute in the first equation, then we get

$$0 = -2\alpha_0 + \alpha_1 + (3/2)\alpha_1$$

or

$$\alpha_1 = (4/5)\alpha_0.$$

Thus, the steady state vector is

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_{1,2}, \alpha_{2,1}) = (\alpha_0, (4/5)\alpha_0, (3/5)\alpha_0, (4/5)\alpha_0, (3/10)\alpha_0).$$

Since these must add up to 1, we have

$$\alpha_0 = \frac{1}{1 + (4/5) + (3/5) + (4/5) + (3/10)} = \frac{10}{10 + 8 + 6 + 8 + 3} = \frac{10}{35}$$

and the overall steady-state probability vector is

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_{1,2}, \alpha_{2,1}) = (10/35, 8/35, 6/35, 8/35, 3/35).$$

Hence, the desired probability that printer 1 is working is

$$\alpha_0 + \alpha_2 = (10/35) + (6/35) = (16/35).$$

We can check our computations by verifying that indeed $\alpha Q = 0$.

(f) (4 points) Let $N(t)$ count the number of instants in the interval $[0, t]$ that a repair is completed, leaving both printers working. What kind of stochastic process is $\{N(t) : t \geq 0\}$?

These times constitute an embedded renewal process for the process, because everything starts over at these times. That is, these times are necessarily independent and identically distributed (i.i.d.). Hence, $\{N(t) : t \geq 0\}$ is a renewal counting process. Since the times are *not* exponentially distributed, it is not a Poisson process.

(g) (BONUS 4 points) Let T be the random time between successive instants that a a repair is completed, leaving both printers working. What is the expected value $E[T]$?

This is a relatively tricky question. It requires some thinking. We can invoke the renewal reward theorem associated with an alternating renewal process. Each such interval starts off with an exponentially distributed interval of expected length 1/2 during which both machines are working. Call that the up time. The down time is the remaining portion of the interval, until both printers are again working for the first time. Clearly, $E[T] = E[U] + E[D]$. Since we already know $E[U]$, it suffices to find $E[D]$.

To find $E[D]$, we use the steady-state probability vector that we have computed. We know that the probability that both machines are working is $\alpha_0 = 10/35 = 2/7$ from parts (c)-(e). On the other hand, by the renewal reward theorem, we know that

$$\alpha_0 = \frac{E[U]}{E[U] + E[D]},$$

where $E[U] = 1/2$. So we can solve for $E[D]$:

$$\frac{2}{7} = \frac{1/2}{(1/2) + E[D]}.$$

Hence, we get

$$E[D] = 5/4.$$

Finally,

$$E[T] = E[U] + E[D] = (1/2) + (5/4) = \frac{7}{4} = 1.75 \quad \text{weeks.}$$

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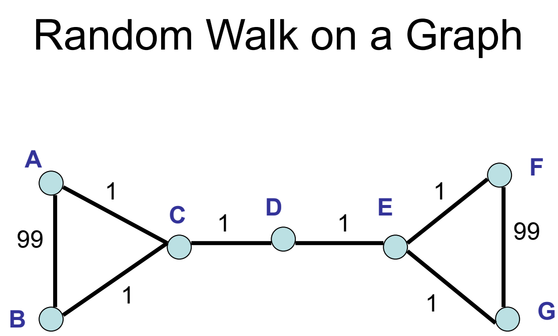
3. Random Walk on a Graph (25 points)

Consider the graph shown in the figure above. There are 7 nodes, labeled with capital letters and 8 arcs connecting some of the nodes. On each arc is a numerical weight. Six of the arcs have weight 1, while two of the arcs have weight 99.



Consider a random walk on this graph, where we move randomly from node to node, always going to a neighbor, via a connecting arc. Let each move be to one of the current node's neighbors, with a probability proportional to the weight on the connecting arc, independent of the history prior to reaching the current node. Thus the probability of moving from node A to node C in one step is $1/(1+99) = 1/100$, while the probability of moving from node C to node A in one step is $1/(1+1+1) = 1/3$. Let X_n be the node occupied after the n^{th} step of the random walk. Suppose that $X_0 = A$.

(a) (3 points) What is the probability of going from node A back to to node A in two steps?

$$\begin{aligned} P^{(2)}_{A,A} &= P_{A,B}P_{B,A} + P_{A,C}P_{C,A} = \left(\frac{99}{99+1}\right)\left(\frac{99}{99+1}\right) + \left(\frac{1}{99+1}\right)\left(\frac{1}{3}\right) \\ &= \frac{(99)^2}{(100)^2} + \frac{1}{3(100)} = 0.9801 + 0.003333 = 0.9834 \end{aligned}$$

(Last arithmetic steps in the second line not required.)

(b) (3 points) What is the probability of going from node A back to to node A in three steps?

$$P^{(3)}_{A,A} \quad = \quad P_{A,B}P_{B,C}P_{C,A} + P_{A,C}P_{C,B}P_{B,A} = \frac{2(99)}{3(100)^2}\frac{198}{30,000} = 0.0066$$

(Last arithmetic steps not required.)

(c) (7 points) Let π_A be the long-run proportion of moves ending in the node A. What is π_A?

Here we recognize that this is the random walk on a weighted graph, as discussed in §4.8 on reversibility in DTMC's. We know that π_A is the sum of the weights out of A divided by the sum over all nodes of the sum of weights out of that node. Thus

$$\pi_A = \frac{99+1}{4(99+1)+8} = \frac{100}{408} = 0.245.$$

(Last arithmetic steps not required.)

(d) (4 points) Starting from node A, what is the expected number of steps required to return to node A?

This expected time is the reciprocal of π_A,

$$\frac{1}{\pi_A} = \frac{408}{100} = 4.08$$

(e) (4 points) Let $T_{A,D}$ be the first passage time from node A to node D, and similarly for other nodes. Is $E[T_{A,D}] > E[T_{F,D}]$? Justify your answer.

No, we instead have **equality**:

$$E[T_{A,D}] = E[T_{F,D}].$$

This is easily explained by **symmetry**. The model is the same if we select the model about a vertical line through node D. That is, we map the state vector (A, B, C, D, E, F, G) into the new state vector (F, G, E, D, C, A, B). There are other symmetries here as well. By similar reasoning, we also have $E[T_{A,D}] = E[T_{B,D}]$. In this case we can reflect the graph about the horizontal axis, through nodes C, D and E. We then change the state vector (A, B, C, D, E, F, G) into the new states (B, A, C, D, E, G, F).

(f) (4 points) Give an expression (not the numerical value) for the expected number of visits to node B, starting from node A, before visiting node F.

This part is answered by applying the **absorbing theory of DTMC's**, as discussed in the lecture notes of September 20. We make node F an absorbing state. Then the remaining 7 nodes become transient states. We then put the transition matrix in canonical form, which we can denote by

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix},$$

where the first row is for the single absorbing state, I is a 1×1 identity matrix, giving the transition probabilities among the absorbing states, which is only the single state F, 0 is a 1×7 matrix of 0's giving the transition probabilities from the absorbing state to the transient states, R is a 7×1 transition matrix giving the one-step probabilities of being absorbed in the absorbing state starting in each of the 7 transient states, and Q is the 7×7 square transition matrix among the 7 transient states (all states except F). The expected value desired is $N_{A,B}$, which is the (A, B) entry of the square matrix

$$N \equiv (I - Q)^{-1},$$

corresponding to row A and column B.

(g) (BONUS 4 points) Compute the expected number of visits to node B, starting from node A, before visiting node C. (Hint: set up a simple recursion.)

We can apply the previous part, after changing the absorbing state from F to C. However, if we make C absorbing and start from A, then we never can reach the states D, E, F and G if we start from A, because we will be absorbed in C. It suffices to consider only the states A, B and C. It suffices to look at

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix},$$

where the first row is the single absorbing state C, I is a 1×1 identity matrix, giving the transition probabilities among these (this one) absorbing states, 0 is a 1×2 matrix of 0's giving the transition probabilities from the absorbing state to the two transient states A and B, R is a 2×1 transition matrix giving the one-step probabilities of being absorbed in the absorbing state starting in each of the 2 transient states, and Q is the 2×2 square transition matrix among the 2 transient states. The expected value desired is $N_{A,B}$, which is the (A, B) entry of the square matrix

$$N \equiv (I - Q)^{-1},$$

corresponding to row A and column B.

But we can use even more elementary reasoning. Let $N_{A,B}$ be the expected total number of visits to B starting in A. To have positive expected value, we must get to B in the first transition, which occurs with probability $99/(99 + 1) = 0.99$. Then to get more contribution, we must return to A, which again occurs with probability $99/(99 + 1) = 0.99$, after which we start over. (Recall Example 3.13 on p. 110.) Thus, we can write down the recursion:

$$N_{A,B} = (0.99)\left(1 + 0.99 N_{A,B}\right).$$

We can then solve for $N_{A,B}$, getting

$$N_{A,B} = \frac{0.99}{1 - (0.99)^2} = \frac{0.99}{1 - 0.9801} = \frac{0.99}{0.0199} \approx 49.74$$

(Last arithmetic steps not required.)

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4. Ten Independent Stocks: Two Investment Strategies (25 points)

You have decided to invest \$800 by buying ten shares at time 0 of stocks initially priced at \$80 per share. Assume that the 10 different stock prices evolve independently over time, with the price of stock j evolving according to the model

$$S_j(t) \equiv 80 + 2.5t + 5B_j(t), \quad t \geq 0,$$

where $\{B_j(t) : t \geq 0\}$ is a standard (drift zero, unit variance) Brownian motion (BM) for each j, with the ten different BM's being stochastically independent.

(a) (2 points) Suppose that you employ a **focused investment strategy** and buy 10 shares of stock 1 at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[10S_1(4)]$ and $Var(10S_1(4))$?

$$\begin{aligned} E[10S_1(4)] &= 10E[80 + (2.5 \times 4) + 5B_1(4)] = 10(80 + 10 + 5E[B_1(4)] \\ &= 10(80 + 10 + (5 \times 0)] = 10(90) = 900 \end{aligned} \tag{4}$$

$$\begin{aligned} Var[10S_1(4)] &= (10)^2 Var(80 + (2.5 \times 4) + 5B_1(4)) = 100Var(5B_1(4)) \\ &= 100 \times 25 \times Var(B(4)) = 100 \times 25 \times 4 = 10,000 \end{aligned} \tag{5}$$

(b) (2 points) With the focused investment strategy in part (a), what is the probability that you will have made a profit? That is, what is the probability that $P(10S_1(4) > 800)$?

By part (a), $10S_1(4) \stackrel{\mathrm{d}}{=} N(900, 10^4)$, i.e. $10S_1(4)$ is normally distributed with the mean and variance above. So that

$$\begin{aligned} P(10S_1(4) > 800) &= P\left(\frac{10S_1(4) - E[10S_1(4)]}{\sqrt{Var(10S_1(4))}} > \frac{800 - E[10S_1(4)]}{\sqrt{Var(10S_1(4))}}\right) \\ &= P\left(N(0,1) > \frac{800 - 900}{100}\right) = P\left(N(0,1) > -1\right) \\ &= 1 - P\left(N(0,1) \leq 1\right) \approx 1 - 0.16 = 0.84 \end{aligned} \tag{6}$$

(c) (2 points) Suppose that, instead, you decide to employ a **diversified investment strategy** and buy 1 share of each of the 10 different stocks at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[S_1(4) + \cdots + S_{10}(4)]$ and $Var(S_1(4) + \cdots + S_{10}(4))$?

The mean is the same as in part (a), but the variance is less. The variance of the sum of independent random variables is the sum of the variances, whereas $Var(cX) = c^2 Var(X)$. In particular,

$$\begin{aligned} E[S_1(4) + \cdots + S_{10}(4)] &= 10E[S_1(4)] = 10(80 + 10 + 5E[B_1(4)] \\ &= 10(80 + 10 + (5 \times 0)] = 10(90) = 900 \end{aligned} \tag{7}$$

$$\begin{aligned} Var[S_1(4) + \cdots + S_{10}(4)] &= 10Var(80 + (2.5 \times 4) + 5B_1(4)) = 10Var(5B_1(4)) \\ &= 10 \times 25 \times Var(B(4)) = 10 \times 25 \times 4 = 1,000 \end{aligned} \tag{8}$$

So the variance is 10 tens smaller than in part (a)!

(d) (2 points) With the alternative investment scheme in part (c), what is the probability that you will have made a profit? That is, what is the probability that $P(S_1(4)+\cdots+S_{10}(4) > 800)$?

Since the variance is $1,000$ instead of $10,000$, the standard deviation is about $\sqrt{1000} \approx 31$ instead of 100.

By part (c), $S_1(4) + \cdots + S_{10}(4) \stackrel{\mathrm{d}}{=} N(900, 10^3)$, i.e. $S_1(4) + \cdots + S_{10}(4)$ is normally distributed with the mean and variance above. So that

$$\begin{aligned} & P(S_1(4) + \cdots + S_{10}(4) > 800) \\ & \quad = P\left(\frac{S_1(4) + \cdots + S_{10}(4) - E[S_1(4) + \cdots + S_{10}(4)]}{\sqrt{S_1(4) + \cdots + S_{10}(4)}} > \frac{800) - ES_1(4) + \cdots + S_{10}(4)]}{\sqrt{Var(S_1(4) + \cdots + S_{10}(4))}}\right) \\ = \; & P\left(N(0,1) > \frac{800 - 900}{31}\right) \approx P\left(N(0,1) > -3.1\right) \\ = \; & 1 - P\left(N(0,1) \leq 3.1\right) \approx 1 - 0.001 = 0.999 \end{aligned} \tag{9}$$

(e) (5 points) True or false: Indicate whether each of the following five statements is true or false, explaining briefly:

(i) For each of the two investment strategies, the total value of the stock at time $t = 4$ is a random variable with a normal probability distribution.

(ii) The total value of the stock at time $t = 4$ has a probability distribution that is the same for both investment strategies.

(iii) An investor whose sole goal is to maximize his expected return should strongly prefer the diversified investment strategy.

(iv) An investor who wants to achieve the expected return of the focused strategy but minimize his risk, as defined by the probability of suffering a loss over the investment period $[0, 4]$, should strongly prefer the diversified investment strategy.

(v) An investor who wants to maximize the probability that he achieves at least 20% more than the expected value should strongly prefer the focused investment strategy.

The goal here is to consolidate the knowledge gained from the previous detailed parts. For the most part, these questions can be answered given that the previous parts have been done correctly. Even if you made numerical mistakes, you could understand the main idea.

(i) TRUE, both distributions are normal.

(ii) FALSE, the variances are very different.

(iii) FALSE, the expected returns are the same with the two strategies.

(iv) TRUE, as shown by specific calculations above.

(v) TRUE, because the greater variance with the focused strategy increases the likelihood of exceeding a value above the mean. This can be shown by calculations just like those of parts (b) and (d) above. Since 20% of 900 is 180, we compare $P(10S_1(4) > 900+180) = P(10S_1(4) > 1080)$ to $P(S_1(4) + \cdots + S_{10}(4) > 1080)$. By obvious modifications of the calculations above, we see that

$$P(10S_1(4) > 1080) > P(S_1(4) + \cdots + S_{10}(4) > 1080).$$

(f) (4 points) Let T be the first time that the share price of stock 1 either exceeds its expected value by \$20 or falls below its expected value by \$10; i.e., let

$$T \equiv \inf\{t > 0 : S_1(t) - E[S_1(t)] \geq 20 \quad \text{or} \quad S_1(t) - E[S_1(t)] \leq -10\}.$$

What are $E[S_1(T)]$, $P(S_1(T) - E[S_1(T)] = 20)$ and $E[T]$? Briefly explain.

The idea here is to apply martingales and the optional stopping theorem (OST). The stochastic process $\{S_1(t) : t \geq 0\}$ is *not* a martingale, because of the drift term, but the stochastic process $\{S_1(t) - E[S_1(t) : t \geq 0\}$ is a martingale. Indeed, when we subtract the mean, the problem is the same as if we had scaled Brownian motion itself; note that

$$S_1(t) - E[S_1(t)] = 5B_1(t), \quad t \geq 0.$$

Thus, we can apply the OST to get

$$E[S_1(T) - E[S_1(T)]] = E[S_1(0) - E[S_1(0)]] = 0,$$

so that

$$P(S_1(T) - E[S_1(T)] = 20) = 1 - P(S_1(T) - E[S_1(T)] = -10) = \frac{10}{10+20} = \frac{1}{3}.$$

Similarly, $\{(S_1(t) - E[S_1(t))^2 - t : t \geq 0\}$ is a martingale. Thus,

$$E[T] = \frac{10 \times 20}{Var(S(1))} = \frac{10 \times 20}{25} = 8$$

Alternatively, we can reason from the fact that $\{B(t)^2 - t : t \geq 0\}$ is a martingale.

Since $\{S_1(t) - E[S_1(t) : t \geq 0\}$ is a martingale, by the OST

$$E[S_1(T) - 80 - 2.5T] = E[S_1(T)] - 80 - 2.5E[T] = 0,$$

so that

$$E[S_1(T)] = 80 + 2.5E[T] = 100.$$

Problem 4 continued: dependence (in one stock over time and between stocks)

(g) (4 points) What are $E[S_1(3)|S_1(4) = 120]$ and $Var[S_1(3)|S_1(4) = 120]$?

This is essentially homework exercise 10.2, assigned in the last homework. We use the formulas in (10.4) of the book, given in §2 of the formula sheet for BM.

This is the conditional expectation and variance looking backwards, i.e., at time s given the value at time t for $0 < s < t$. First, substitute in and simplify

$$\begin{aligned} E[S_1(3)|S_1(4) = 120] &= E[80 + 2.5(3) + 5B(3)|80 + 2.5(4) + 5B(4) = 120] \\ &= 87.5 + 5E[B(3)|5B(4) = 30] = 87.5 + 5E[B(3)|B(4) = 6] \\ &= 87.5 + 5(4.5) = 87.5 + 22.5 = 110 \end{aligned} \tag{10}$$

For this part, you could reason directly, getting

$$E[S_1(3)|S_1(4) = 120] = 80 + (3/4)(120 - 80) = 80 + 30 = 110.$$

We now turn to the conditional variance.

$$\begin{aligned} Var[S_1(3)|S_1(4) = 120] &= Var[80 + 2.5(3) + 5B(3)|80 + 2.5(4) + 5B(4) = 120] \\ &= Var(5E[B(3)|5B(4) = 30] = 25Var[B(3)|B(4) = 6] \\ &= \frac{25 \times 3 \times (4 - 3)}{4} = \frac{75}{4} = 18.75 \end{aligned} \tag{11}$$

(h) (4 points) What are the expected values: $E[S_1(3)S_1(4)]$ and $E[S_1(4)S_2(4)]$? (The stock price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ are still assumed to be independent in this part.)

For parts (f) and (g), we reduce the analysis to basic facts involving BM $B(t)$. First, recall that, for $0 < s < t$,

$$E[B(s)B(t)] = E[B(s)(B(s) + B(t) - B(s))] = E[B(s)^2] = Var(B(s)) = s.$$

Then observe that

$$\begin{aligned} E[S_1(3)S_1(4)] &= E[(80 + 2.5(3) + 5B(3))(80 + 2.5(4) + 5B(4))] \\ &= E[(87.5 + 5B(3))(90 + 5B(4))] \\ &= (87.5 \times 90) + 450E[B(3)] + 437.5E[B(4)] + 25E[B(3)B(4)] \\ &= (87.5 \times 90) + 0 + 0 + (25 \times (3) = (87.5 \times 90) + 75 \\ &= 7875 + 75 = 7950 \end{aligned} \tag{12}$$

(The last multiplication and final calculation in the final line is not needed.)

Turning to the second question, Since the different stock prices are i.i.d., we have

$$E[S_1(4)S_2(4)] = E[S_1(4)]E[S_2(4)] = (E[S_1(4)])^2 = (90)^2 = 8100.$$

(i) (BONUS 4 points) Suppose that there are stocks for which the stock prices are in fact **stochastically dependent** in various ways. If you could **control the dependence** between the stocks without altering the probability law of each individual stock price process (e.g., by picking the stocks in some clever way), then how could you do better than the diversified investment strategy above? That is, how could you (i) control the dependence and (ii) make an investment strategy in order to reduce the risk (the probability of not making a profit) while leaving the overall expected value of the investment at time 4 (or any other time) unchanged?

The point of this part of the problem is to introduce the idea of **hedging** in finance. (For background, look at "hedge in finance" in Wikipedia.) The idea is to invest in two stocks that are strongly negatively correlated, so that excursions of one above or below the expected value will be matched by excursions in the other in the opposite direction. We look for stocks that have similar expected value behavior, but are as negatively correlated as possible. That is, we want to invest in two stocks that have -1 correlation. Recall that the **correlation** of two random variables is defined as

$$\rho_{X,Y} \equiv Corr(X,Y) \equiv \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}},$$

where

$$Cov(X,Y) \equiv E[(X - E[X])(Y - E[Y]) = E[XY] - E[X]E[Y]$$

is the **covariance**. The correlation can assume values in the interval $[-1, 1]$ and only there. Thus a correlation of -1 is the most negative. That is achieved when $Y = -X$. That is what we are aiming for in the variability part of the stocks. But we want to leave the positive drift unchanged.

The stochastic dependence between two stock price processes $\{S_j(t) : t \geq 0\}$ for $j = 1, 2$ defined in terms of Brownian motions $\{B_j(t) : t \geq 0\}$ for $j = 1, 2$, as in this problem, will have dependence determined by the dependence in the underlying Brownian motions. We use this structure to artificially **control the dependence**. We can achieve perfectly negative dependence (the best possible) by letting

$$B_2(t) = -B_1(t), \quad t \geq 0.$$

Notice that the stochastic process $\{-B_1(t) : t \geq 0\}$ has the same Brownian motion probability law as a stochastic process as the original Brownian motion $\{B_1(t) : t \geq 0\}$, but of course

$$B_1(t) + B_2(t) = 0 \quad \text{for all} \quad t \geq 0.$$

As before, for $j = 1, 2$, we let

$$S_j(t) \equiv 80 + 2.5t + 5B_j(t), \quad t \geq 0,$$

but where $B_2(t) \equiv -B_1(t)$, $t \geq 0$, as above.

Our new **hedging strategy based on dependence** involves buying equal amounts of the two stocks. We would thus buy 5 shares of stock $\{S_1(t) : t \geq 0\}$ and 5 shares of $\{S_2(t) : t \geq 0\}$. Since the Brownian motions have the proper probability distribution, these processes individually have the correct probability law. However, we achieve our objective, because

$$5S_1(t) + 5S_2(t) = 800 + 25t, \quad t \geq 0,$$

without any variability at all. Note that

$$E[5S_1(t) + 5S_2(t)] = 800 + 25t \quad \text{and} \quad Var(5S_1(t) + 5S_2(t)) = 0 \quad \text{for all} \quad t \geq 0.$$

so that, for $t = 4$,

$$E[5S_1(4) + 5S_2(4)] = 900 \quad \text{and} \quad Var(5S_1(4) + 5S_2(4)) = 0 \quad \text{for all} \quad t \geq 0.$$

We have kept the same mean, but reduced the variance to 0.

In practice, something like this is achieved with financial instruments that are very negatively correlated. Usually some of the expected value must be sacrificed in order to successfully hedge.