IEOR 3106: Final Exam, December 15, 2013

SOLUTIONS

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. You may keep the exam itself. Solutions will eventually be posted on line.

1. Forecasting the Weather (20 **points)**

Consider the following probability model of the weather over successive days. First, suppose that on each day we can specify if the weather is rainy or dry. Suppose that the probability that it will be rainy on any given day is a function of the weather on the previous two days. If it was rainy both yesterday and today, then the probability that it will be rainy tomorrow is 0.7. If it was dry yesterday, but rainy today, then the probability that it will be rainy tomorrow is 0.5. If it was rainy yesterday, but dry today, then the probability that it will be rainy tomorrow is 0.4. If it was dry both yesterday and today, then the probability that it will be rainy tomorrow is 0.2. Let X_n be the weather on day n.

First, we recognize that this is Example 4.4 in the textbook.

(a) Calculate the conditional probability that it rains tomorrow but is dry on the next two days, given that it rained both yesterday and today. (5 **points,** 1 **for product of three,** 1 **for correct first term**)

$$\begin{aligned} P(X_{n+1} = R, X_{n+2} = D, X_{n+3} = D | X_{n-1} = R, X_n = R) &= P(X_{n+1} = R | X_{n-1} = R, X_n = R) \\ \times P(X_{n+2} = D | X_n = R, X_{n+1} = R) \times P(X_{n+3} = D | X_{n+1} = R, X_{n+2} = D) \\ = (0.7)(0.3)(0.6) = 0.126 \end{aligned}$$

(b) (5 points) Is the stochastic process $\{X_n : n \geq 0\}$ a Markov chain? Why or why not? If not, construct an alternative finite-state stochastic process for this problem that is a Markov chain. (5 **points,** 2 **for saying "no" and** 3 **for a correct DTMC**)

No, the stochastic process $\{X_n : n \geq 0\}$ is *not* a Markov chain. It fails to have the Markov property. The probability of a future event, e.g., the weather tomorrow, conditional on present and past states does *not* depend only on the present state.

We now develop a DTMC. We let the state be (X_{n-1}, X_n), combining the states of X_n on days $n-1$ and n. Thus there are four possible states instead of two. The DTMC then transitions from the state (X_{n-1}, X_n) to the state (X_n, X_{n+1}). Note that part of the new stat is determined by the previous state. The model is the transition matrix

$$P = \begin{array}{c} 1 \equiv RR \\ 2 \equiv DR \\ 3 \equiv RD \\ 4 \equiv DD \end{array} \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix} ,$$

where the columns are labeled in the same way, and the same order, as the rows.

(c) With your Markov chain model in part (b), calculate the long-run proportion of days that are rainy. (The proper equation is good, but a numerical answer is better.) (5 **points,** 2 **for** $\pi = \pi P$ **and** 3 **for a full answer**)

We solve $\pi = \pi P$ with the 4×4 transition matrix in part (b). That gives the limiting probability

$$\lim_{n \to \infty} \{P(((X_{n-1}, X_n) = j)\},$$

where j is one of the four states. To get the desired long-run probability we must add $\pi_{(R,R)} + \pi_{(D,R)} = \pi_{(R,R)} + \pi_{(R,D)}$. (Equality holds because we can look at either the limit of (X_{n-1}, X_n) and sum over the possible values of X_{n-1} to get the marginal distribution of X_n or the limit of (X_n, X_{n+1}) and sum over the possible values of X_{n+1} to get the marginal distribution of X_n.)

When solving $\pi = \pi P$, we get the equations:

$$\begin{aligned}
0.7\pi_1 + 0.5\pi_2 &= \pi_1 \\
0.4\pi_3 + 0.2\pi_4 &= \pi_2 \\
0.3\pi_1 + 0.5\pi_2 &= \pi_3 \\
0.6\pi_3 + 0.8\pi_4 &= \pi_4 \\
\pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1
\end{aligned}$$

From equation 1, we get $\pi_1 = (5/3)\pi_2$. Then, using this in equation 3, we get $\pi_3 = 0.3(5/3)\pi_2 + 0.5\pi_2 = \pi_2$. Hence, $\pi_2 = \pi_3$ From equation 4, we get $\pi_4 = 3\pi_3$. Combining the last two equations, we get $\pi_4 = 3\pi_2$.

From the final equation, we then get $\pi_2 = 3/20$. Thus,

$$\pi \equiv (\pi_1, \pi_2, \pi_3, \pi_4) = (5/20, 3/20, 3/20, 9/20).$$

Thus the long run proportion of days that are rainy is

$$\pi_1 + \pi_2 = \pi_1 + \pi_3 = \frac{8}{20} = \frac{2}{5}.$$

(d) Which is larger: (i) the long-run proportion of days that it was rainy yesterday and dry today or (ii) the long-run proportion of days that it was dry yesterday and rainy today? (5 **points,** 2 **for right answer and** 3 **for a demonstration**)

These must be equal in general. Here is the reasoning: The long-run proportion of days that it was rainy yesterday and dry today is $\pi_{R,D}$; the long-run proportion of days that it was dry yesterday and rainy today $\pi_{D,R}$; the long-run proportion of days that it was rainy yesterday is $\pi_{R,D} + \pi_{R,R}$; the long-run proportion of days that it was rainy today is $\pi_{D,R} + \pi_{R,R}$. Since the long-run proportion of days that it was rainy yesterday necessarily equals the long-run proportion of days that it was rainy today, we have

$$\pi_{R,D} + \pi_{R,R} = \pi_{D,R} + \pi_{R,R}.$$

Subtracting $\pi_{R,R}$ from both sides we get equality:

$$\pi_{R,D} = \pi_{D,R}.$$

However, from your answer in part (c), you can see that this property holds in this example. Both are 3/20.

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2. A Computer with Three parts (20 points)

A computer has three critical parts, each of which is needed for the computer to work. The computer runs continuously as long as the three required parts are working. The three parts have mutually independent exponential lifetimes before they fail. The expected lifetime of parts 1, 2 and 3 are 10 weeks, 20 weeks and 30 weeks, respectively. When a part fails, the computer is shut down and an order is made for a new part of that type. When the computer is shut down (to order a replacement part), the remaining two working parts are not subject to failure. The time to replace part 1 is exponentially distributed with mean 1 week; the time to replace part 2 is uniformly distributed between 1 week and 3 weeks; and the time to replace part 3 has a gamma distribution with mean 3 weeks and standard deviation 10 weeks.

(a) Assuming that all parts are initially working, what is the expected time until the first part fails?

Let T be the time until the first failure. Then T is exponential with rate equal to the sum of the rates; see Chapter 5; i.e.,

$$ET = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{(1/10) + (1/20) + (1/30)} = \frac{1}{(11/60)} = \frac{60}{11} = 5.45 \text{ weeks} .$$

(b) What is the probability that part 1 is the first part to fail?

Let N be the index of the first part to fail. Since the failure times are mutually independent exponential random variables (see Chapter 5),

$$P(N = 1) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{(1/10)}{(1/10) + (1/20) + (1/30)} = \frac{6}{11} = 0.545 .$$

(c) What is the long-run proportion of time that the computer is working?

Now for the first time we need to consider the random times it takes to get the replacement parts. Actually these distributions beyond their means do not affect the answers to the questions asked here. Only the means matter here. Use elementary renewal theory. The successive times that the computer is working and shut down form an alternating renewal process. Or, equivalently, apply renewal reward processes, as in Section 7.4: Look at the expected reward per cycle divided by the expected length of a cycle. Let a reward be earned at rate 1 whenever the computer is working. Let T be a time until a failure (during which the computer is working) and let D be a down time. A cycle is $T + D$. Then the long-run proportion of time that the computer is working is

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t R(s)\, ds = \lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{N(t)} T_i = \frac{ET}{ET + ED},$$

where $N(t)$ is the renewal counting process counting the number of complete cycles up to time t. (This ignores the last cycle in process at time t, but the contribution of it is asymptotically negligible.)

By part (a) above, $ET = 60/11$. It suffices to find ED. To find ED, we consider the three possibilities for the part that fails:

$$\begin{aligned} ED &= P(N=1)E[D|N=1] + P(N=2)E[D|N=2] + P(N=3)E[D|N=3] \\ &= (6/11)E[D|N=1] + (3/11)E[D|N=2] + (2/11)E[D|N=3] \\ &= (6/11)1 + (3/11)2 + (2/11)3 \\ &= 18/11\ . \end{aligned}$$

Hence,

$$ET/(ET + ED) = \frac{(60/11)}{(60/11) + (18/11)} = \frac{60}{78} = \frac{30}{39} \approx 0.769$$

(d) Suppose that new parts of type 1 each cost \$50; new parts of type 2 each cost \$100; and new parts of type 3 each cost \$400. What is the long-run average cost of replacement parts per week?

This is yet another application of the renewal reward theorem. The cycle is the same as before, but now we have a new reward R. Now

$$\begin{aligned} \frac{ER}{EC} &= \frac{ER}{ET + ED} \\ &= \frac{P(N=1)50 + P(N=2)100 + P(N=3)400}{78/11} \\ &= \frac{(6/11)50 + (3/11)100 + (2/11)400}{78/11} \\ &= \frac{(300/11) + (300/11) + (800/11)}{78/11} \\ &= \frac{1400/11}{78/11} = \frac{1400}{78} \approx 17.95 \text{ dollars per week} \end{aligned}$$

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3. Investment Strategies for Brownian Stocks(25 points)

You have decided to invest \$4000 by buying 40 shares of stocks at time 0, each initially priced at \$100 per share.

Part a. Four Independent Stocks (16 points)

Suppose that you are considering the 4 different stocks, whose prices evolve independently over time, with the price of stock j evolving according to the model

$$S_j(t) \equiv 100 + 2.5t + 10B_j(t), \quad t \geq 0, \tag{1}$$

where $\{B_j(t) : t \geq 0\}$ is a standard (drift zero, unit variance) Brownian motion (BM) for each j, with the four different BM's being stochastically independent.

(a) (2 points) Suppose that you employ a **focused investment strategy** and buy 40 shares of stock 1 at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[40S_1(4)]$ and $Var(40S_1(4))$?

$$\begin{aligned} E[40S_1(4)] &= 40E[100 + (2.5 \times 4) + 10B_1(4)] = 40(100 + 10 + 10E[B_1(4)] \\ &= 40(100 + 10 + (10 \times 0)] = 40(110) = 4400 \end{aligned}$$

$$\begin{aligned} Var[40S_1(4)] &= (40)^2 Var(100 + (2.5 \times 4) + 10B_1(4)) = 1600Var(10B_1(4)) \\ &= 1600 \times 100 \times Var(B(4)) = 1600 \times 100 \times 4 = 640,000 = (800)^2 \end{aligned}$$

(b) (2 points) With the focused investment strategy in part (a), what is the probability that you will have made a profit? That is, what is the probability that $P(40S_1(4) > 4000)$?

By part (a), $40S_1(4) \stackrel{\text{d}}{=} N(4400, (800)^2)$, i.e. $40S_1(4)$ is normally distributed with the mean and variance above. So that

$$\begin{aligned} P(40S_1(4) > 4000) &= P\left(\frac{40S_1(4) - E[40S_1(4)]}{\sqrt{Var(40S_1(4))}} > \frac{4000 - E[40S_1(4)]}{\sqrt{Var(40S_1(4))}}\right) \\ &= P\left(N(0,1) > \frac{4000 - 4400}{800}\right) = P\left(N(0,1) > -0.5\right) \\ &= P\left(N(0,1) \leq 0.5\right) \approx 0.6915 \end{aligned}$$

(c) (4 points) Let T be the first time t that the share price of stock 1 either exceeds its expected value at time t by \$20 or falls below its expected value by \$10; i.e., let

$$T \equiv \inf\{t > 0 : S_1(t) - E[S_1(t)] \geq 20 \quad \text{or} \quad S_1(t) - E[S_1(t)] \leq -10\}.$$

What is the probability $P(S_1(T) = E[S_1(T)] + 20)$ and what is $E[T]$? Briefly explain.

The idea here is to apply martingales and the optional stopping theorem (OST). The stochastic process $\{S_1(t) : t \geq 0\}$ is *not* a martingale, because of the drift term, but the stochastic process $\{S_1(t) - E[S_1(t) : t \geq 0\}$ is a martingale. Indeed, when we subtract the mean, the problem is the same as if we had scaled Brownian motion itself; note that

$$S_1(t) - E[S_1(t)] = 10B_1(t), \quad t \geq 0.$$

Thus, we can apply the OST to get

$$E[S_1(T) - E[S_1(T)]] = E[S_1(0) - E[S_1(0)]] = 0,$$

so that

$$P(S_1(T) - E[S_1(T)] = 20) = 1 - P(S_1(T) - E[S_1(T)] = -10) = \frac{10}{10+20} = \frac{1}{3}.$$

We see that the hitting time of $+20$ or -10 by $10B(t)$ is distributed the same as the htting time of $+2$ or -1 by $B(t)$. Hence, the formula is

$$E[T] = 2 \times 1 = 2.$$

However, we can also reason directly, repeating the argument in the lecture notes of 12/3/13. In [articular, since $\{B(t)^2 - t : t \geq 0\}$, is the quadratic MG, $\{\sigma B(t)^2 - \sigma^2 t) : t \geq 0\}$ is a MG, so that $\{(S_1(t) - E[S_1(t))^2 - 100t : t \geq 0\}$ is a martingale. Thus,

$$E[T] = \frac{10 \times 20}{100} = \frac{10 \times 20}{100} = 2.$$

That is the same answer as we got above.

(d) (2 points) Suppose that, instead, you decide to employ a **diversified investment strategy** and buy 10 shares of each of the 4 different stocks at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[10S_1(4) + \cdots + 10S_4(4)]$ and $Var(10S_1(4) + \cdots + 10S_4(4))$?

The mean is the same as in part (a), but the variance is less. The variance of the sum of independent random variables is the sum of the variances, whereas $Var(cX) = c^2 Var(X)$. In particular,

$$\begin{aligned} E[10S_1(4) + \cdots + 10S_4(4)] &= 40E[S_1(4)] = 40(100 + 10 + 10E[B_1(4)] \\ &= 40(100 + 10 + (10 \times 0)] = 40(110) = 4400 \end{aligned}$$

$$\begin{aligned} Var[10S_1(4) + \cdots + 10S_4(4)] &= (4 \times 100)Var(100 + (2.5 \times 4) + 10B_1(4)) = 400Var(10B_1(4)) \\ &= 400 \times 100 \times Var(B(4)) = 400 \times 100 \times 4 = 160,000 = (400)^2 \end{aligned}$$

So the variance is 4 tens smaller than in part (a).

(e) (2 points) With the alternative investment scheme in part (c), what is the probability that you will have made a profit? That is, what is the probability that $P(10S_1(4) + \cdots + 10S_4(4) > 4000)$?

Since the variance is $(400)^2$ instead of $(800)^2$, the standard deviation is 400 instead of 800.

By part (c), $10S_1(4) + \cdots + 10S_4(4) \stackrel{\mathrm{d}}{=} N(4400, (800)^2)$, i.e. $10S_1(4) + \cdots + 10S_4(4)$ is normally distributed with the mean and variance above. So that

$$
\begin{aligned}
&P(10S_1(4) + \cdots + 10S_4(4) > 4000) \\
&\quad = P\left(\frac{10S_1(4) + \cdots + 10S_4(4) - E[10S_1(4) + \cdots + 10S_4(4)]}{\sqrt{10S_1(4) + \cdots + 10S_4(4)}} > \frac{4000) - E[10S_1(4) + \cdots + 10S_4(4)]}{\sqrt{Var(10S_1(4) + \cdots + 10S_4(4))}}\right) \\
= \;& P\left(N(0,1) > \frac{4000 - 4400}{400}\right) \approx P\left(N(0,1) > -1.0\right) \\
= \;& P\left(N(0,1) \le 1.0\right) \approx 0.84
\end{aligned}
$$

The probability of making a profit has gone up from 0.69 to 0.84 from part (b).

(f) (4 points) True or false: Indicate whether each of the following five statements is true or false, explaining briefly:

(i) The total value of the stock at time $t = 4$ has a probability distribution that is the same for both investment strategies.

No, FALSE, we just saw in parts (a) and (c) that the variance is different in the two cases.

(ii) An investor whose sole goal is to maximize his expected return should strongly prefer the diversified investment strategy.

No, FALSE. Since the expected value is the same, such an investor would be indifferent between these investment options. Of course, the objective of this investor is not so reasonable.

(iii) An investor who wants to achieve the largest possible expected return but minimize his risk, as defined by the probability of suffering a loss over the investment period $[0, 4]$, should strongly prefer the diversified investment strategy.

Yes, TRUE. We have shown that in parts (b) and (d) above.

(iv) An investor who wants to maximize the probability that he achieves at least 20% more than the expected value should strongly prefer the focused investment strategy.

Yes, TRUE. The higher variance with the focused strategy makes the probability of large values more likely. But that is achieved at increased risk. The probability of a loss increases as well.

Part b. Three Other Stocks Dependent on the Initial Group (9 points)

Suppose that, in addition to the four Brownian stocks specified in (1), we also have three other stocks with

$$S_j(t) \equiv 100 + 2.5t + 10X_j(t), \quad t \ge 0, \tag{2}$$

for $j = 5$, 6 and 7, where the prices of these stocks depend on the prices of the first three stocks in (1). Specifically, suppose that $X_j(t)$ in (2) are related to the Brownian motions $B_j(t)$ in (1) by

$$\begin{aligned} X_5(t) &\equiv B_1(t) - B_3(t), \quad t \geq 0, \\ X_6(t) &\equiv B_2(t) - B_1(t), \quad t \geq 0, \\ X_7(t) &\equiv 2B_3(t) - 2B_2(t), \quad t \geq 0. \end{aligned}$$

(g) (3 points) Suppose that you employ a **focused investment strategy** and buy 40 shares of stock 5 at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[40S_5(4)]$ and $Var(40S_5(4))$?

Mostly, this is just like part (a), but we have to be careful with $X_5(t)$. Notice that $\{-B(t); t \geq 0\}$ is distributed the same as $\{B(t) : t \geq 0\}$. Hence $\{B_1(t) - B_3(t) : t \geq 0\}$ is distributed the same as $\{B_1(t) + B_3(t) : t \geq 0\}$, which in turn is distributed the same as $\{\sqrt{2}B(t) : t \geq 0\}$. For this specific problem, we write

$$\begin{aligned} E[40S_5(4)] &= 40E[100 + (2.5 \times 4) + 10X_5(4)] = 40(100 + 10 + 10E[B_1(4) - B_3(4)] \\ &= 40(100 + 10 + (10 \times 0)] = 40(110) = 4400 \end{aligned}$$

$$\begin{aligned} Var[40S_5(4)] &= (40)^2 Var(100 + (2.5 \times 4) + 10X_1(4)) = 1600 Var(10(B_1(4) - B_3(4)) \\ &= 1600 \times 100 \times (Var(B_1(4)) + Var(B_3(4))) \\ &= 1600 \times 100 \times 8 = 1,280,000 = (800\sqrt{2})^2 \end{aligned}$$

Note that the variance of $B_1(4) - B_3(4)$ is the same as $2Var(B_1(4))$. The variance of $X_5(4)$ is larger than the variance of $B_1(4)$.

(h) (3 points) Find an investment strategy for investing your 4000 in these 7 stocks in (1) and (2) that maximizes the probability of your making at least 1000 profit at time $t = 4$.

We want to be risk seeking. All the strategies yield the same expected value 4400. We want to make the probability as large as possible that we make 600 more than our expected profit of 400. To do that, we want to make the variance as large as possible. That is done by using a focused investment strategy and investing all 4000 in shares of stock 7.

To see this, note that

$$\begin{aligned} Var[40S_7(4)] &= (40)^2 Var(100 + (2.5 \times 4) + 10X_7(4)) = 1600 Var(10(2B_3(4) - 2B_2(4)) \\ &= 1600 \times 400 \times (Var(B_3(4)) + Var(B_2(4))) \\ &= 1600 \times 400 \times 8 = 5,120,000 = (1600\sqrt{2})^2 \end{aligned}$$

Note that the variance of $2B_3(4) - 2B_2(4)$ is the same as $8Var(B_1(4))$. Just as we saw that the variance of $X_5(4)$ is larger than the variance of $B_1(4)$ in part (g), we now see that the variance of $X_7(4)$ is even larger still.

(i) (3 points) Find an investment strategy for investing your 4000 in these 7 stocks that achieves *both* the highest expected profit and the lowest risk, as specified by the probability of not making a profit at time $t = 4$, i.e., the probability that the value at time $t = 4$ is less than \$4000.

Now we want to make the variance as small as possible. Since we always have the same expected value, there is no concern about the mean. But, in this case (with these assumptions), the variance can be reduced to 0, so that there is no risk at all! We should buy 16 shares of S_5, 16 shares of S_6 and 8 shares of S_7, costing us $1600 + 1600 + 800 = 4000$, our amount to invest. So we have expected value $16E[S_5(4)] + 16E[S_6(4)] + 8E[S_7(4)] = 4400$, but there is no randomness left at all, because

$$16X_5(t) + 16X_6(t) + 8X_7(t) = 16(B_1(t) - B_3(t)) + 16(B_2(t) - B_1(t)) + 8(2B_3(t) - 2B_1(t)) = 0$$

Hence,

$$16S_5(t) + 16S_6(t) + 8S_7(t) = 4000 + 100t, \quad t \geq 0,$$

so that

$$P(16S_5(4) + 16S_6(4) + 8S_7(4) = 4400) = 1.$$

That is, with this strategy, we eliminate all risk. We have "the perfect hedge."

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4. An Airport Security Check (35 points)

An airport security check for departing passengers has been designed with two inspection stations. At each inspection station, passengers are processed one at a time in order of arrival at that station. There is ample waiting space at each station. All departing passengers go through the first (standard) stage of inspection. However, only those passengers that fail the first stage of inspection go to the second stage. There is a more elaborate inspection at the second station.

Suppose that passengers arrive at station 1 according to a Poisson process with rate 4 per minute; Suppose that the processing times at the stations are independent exponentially distributed random variables. Let the mean processing times be 10 *seconds* at station 1 and 10 *minutes* at station 2. Suppose that each successive passenger fails first-stage inspection, and thus requires second stage inspection, with probability p (independent of the history prior to the passenger).

We remind you that this question relates to reversibility of CTMC's and queueing networks, discussed in the lecture for October 29. See those lecture notes for pointers to the CTMC notes and the book. However, the first part only concerns Poisson processes and exponential random variables. The first part is about Chapter 5 in the book. The first part is about the two lectures of October 8 and 10 (the post office and gone fishing). Similarly, the second part only concerns the first station, which involves a birth and death process. Hence, the second part is covered by the lecture of October 17 (the barbershop). We are expecting that everybody should be able to do the first two parts, while the third part is more challenging. The third part provides a chance to demonstrate mastery of the material.

The First Station Starting Empty (9 points, 3 each)

Consider arrivals to the first inspection station at the beginning of the day, starting empty.

(a) Let T_k be the the arrival time of the k^{th} passenger at the first inspection station. What are the mean and variance: $E[T_k]$ and $Var(T_k)$?

Recall that the interarrival times of a Poisson process with rate λ are i.i.d. exponential random variables with mean $1/\lambda$. Recall that the mean of a sum is always the sum of the means. Recall that the variance of a sum of independent random variables is the sum of the variances. Hence,

$$E[T_k] = \frac{k}{4} \quad \text{minutes} \quad \text{and} \quad Var(T_k) = \frac{k}{4^2} = \frac{k}{16}.$$

(b) What is the probability that three passengers arrive at the first inspection station before any inspections have been completed?

Of course, we have to start with the first arrival. Then, twice in a row, we need to have an arrival before a service completion. By the lack of memory property, these are independent experiments. We have two independent experiments involving the minimum of two independent exponential random variables. Hence the answer is

$$\left(\frac{\lambda}{\lambda+\mu}\right)^2 = \left(\frac{4}{4+6}\right)^2 = 0.16$$

We must be careful about the units. We have these numbers because the service rate is 6 per minute.

(c) Suppose that 30 passengers arrive at the first inspection station during the first 4 minutes. What is the probability distribution of the number of these passengers that arrived during the first minute?

We use the conditional uniform property of the Poisson process. Conditional on 30 arrivals in $[0,4]$, these arrival times are distributed the same as 30 i.i.d. uniform random variables in the interval $[0,4]$. Hence, each falls in the interval $[0,1]$ with probability $1/4$. Hence, the answer is a binomial probability, namely,

$$b(k; 30, 1/4) = \left(\frac{30!}{k!(30-k)!}\right)(1/4)^k(3/4)^{30-k}.$$

The Number of Passengers at the First Station (10 points)

(d) Let $Q_1(t)$ be the number of passengers at station 1 at time t. True or false: Indicate whether each of the following statements is true or false (and briefly explain). **(4 points, −2 for each wrong answer, minimum score 0)**

(i) The stochastic process $\{Q_1(t) : t \geq 0\}$ is a Poisson process.
(ii) The stochastic process $\{Q_1(t) : t \geq 0\}$ is a Markov process.
(iii) The stochastic process $\{Q_1(t) : t \geq 0\}$ is a birth and death process.
(iv) There exists a probability vector $\alpha \equiv (\alpha_k : k \geq 0)$ such that $P(Q_1(t) = k) = \alpha_k$ for all $k \geq 0$ and $t \geq 0$ if we set $P(Q_1(0) = k) = \alpha_k$ for $k \geq 0$.

(v) If there exists a stochastic process $\{Q_1(t) : t \geq 0\}$ with $P(Q_1(t) = k) = \alpha_k$ for all $k \geq 0$ and $t \geq 0$, then that stochastic process is a time-reversible Markov process.

(i) FALSE. A Poisson process has nondecreasing sample paths and independent increments, not true for Q_1.

(ii) -(v). TRUE. It is an irreducible BD process. Moreover, it is positive recurrent, because the rate in is less than the rate out. If we do the BD analysis, then we can calculate the steady-state distribution α. All positive irreducible BD processes are time-reversible continuous-time Markov chains (assuming that we work with the stationary version identified in part (iv)). See sections of notes on BD processes and reversible CTMC's.

(e) What is the probability that at some fixed time in steady state there are 4 passengers at station 1, either being inspected or waiting to be inspected? **(4 points)**

Let $Q_1(t)$ be the number at the first station at time t and let $Q_1(\infty)$ be the steady-state number. The number at the first station is a BD process, with constant birth rate and death rate; i.e., the first station is an $M/M/1$ queue. We can apply the formula for BD processes

The steady-state probability is thus

$$\alpha_4 \quad \equiv \quad P(Q_1(\infty) = 4) = \frac{r_4}{\sum_{k=0}^{\infty} r_k} = \frac{\rho_1^4}{\sum_{k=0}^{\infty} \rho_1^k} = (1-\rho_1)\rho_1^4 = (1/3)(2/3)^4 = \frac{2^4}{3^5} = \frac{16}{729},$$

where, in terms of general BD birth and death rates λ_k and μ_k,

$$r_k \equiv \frac{\lambda_0 \times \cdots \times \lambda_{k-1}}{\mu_1 \times \cdots \mu_k} = \frac{\lambda^k}{\mu^k} = \rho_1^k,$$

with λ being the constant arrival rate at station 1 and μ being the constant service rate at station 1, so that (being careful to use the same units for time)

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{4}{6} = 2/3.$$

(There are two possible meanings for the notation λ_1 and μ_1, but they have the same value.)

(f) What is the expected steady-state number of passengers at station 1, either being inspected or waiting to be inspected? **(2 points)**

The answer is the expected value of the geometric distribution derived in part (e). Most of work and credit for part (e). The expected number is thus

$$E[Q_1(\infty)] = \frac{\rho_1}{1-\rho_1} = \frac{2/3}{1/3} = 2,$$

where (being careful to use the same units for time)

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{4}{6} = 2/3.$$

The Numbers of Passengers at the Two Stations (16 points)

For $j = 1, 2$, let $Q_j(t)$ be the number of passengers at station j at time t, $t \geq 0$.

(g) True or false: Indicate whether each of the following statements is true or false (and briefly explain): **(4 points, -1 for each wrong answer, minimum score 0)**

(i) The stochastic process $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ is a Poisson random measure or Poisson process on the plane.

(ii) The stochastic process $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ is a continuous-time Markov chain.

(iii) The stochastic process $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ is a birth and death process.

(iv) For all strictly positive values of the parameter p (the probability of a passenger requiring second-stage inspection), there is stationary version of the stochastic process $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ with $P(Q_1(t) = k_1, Q_2(t) = k_2) = \alpha_{k_1,k_2}$ for all $t \geq 0$, $k_1 \geq 0$ and $k_2 \geq 0$, where $\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \alpha_{k_1,k_2} = 1$.

(v) If there does exist a stationary version of the stochastic process $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ as described in part (iv), then that stationary version with the parameters above is a time-reversible Markov process.

(vi) If there does exist a stationary version of the stochastic process $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ as described in part (iv), then that stationary version has the product form, i.e.,

$$P(Q_1(t) = k_1, Q_2(t) = k_2) = P(Q_1(t) = k_1)P(Q_2(t) = k_2) \quad \text{for all} \quad t, k_1, k_2 \geq 0,$$

where $Q_j(t)$ has the stationary distribution of a birth and death process.

(i) FALSE. A Poisson process has nondecreasing sample paths and independent increments, not true for (Q_1, Q_2).

(ii). TRUE. It is a CTMC.

(iii). FALSE. It is NOT a BD process. The two-dimensional process does not go up and down 1 in each transition. It is not even a multi-dimensional BD process (which we did not discuss), because it has transitions from (i, j) to $(i - 1, j + 1$ when we have a service completion at the first station.

(iv) FALSE. We need p small enough so that the rate into the second station is less than the maximum rate out. This question is asked again in more detail below in part (i). There we specify which values of p are OK.

(v). FALSE. The two-dimensional process is NOT reversible. Not that we can have a transition from state (i, j) to state $(i - 1, j + 1)$ by having a departure from the first station, but we cannot have a transition in the reverse direction.

(vi). TRUE. Even though the two-dimensional process is not reversible, we DO have this desirably property. All of this is discussed in the sections of the CTMC notes on reversibility and open queueing networks. We get the product form here either by exploiting the reversibility of the first station alone or by exploiting Kelly's lemma involving the reverse-time CTMC. The reverse-time CTMC is well defined, corresponding to a two-queue network with time reversed, so that departures are transformed into arrivals. The theory for open queueing networks mostly exploits the reverse-time CTMC, and not the reversibility, which holds for individual stations.

(h) In steady state, what is the probability that exactly 8 passengers complete the first-stage inspection during a given 2 minute interval? Justify your answer. **(3 points)**

Since the number at the the first station is a BD process, and since there is a proper steady-state distribution, the stationary version is time reversible. Since the departure process coincides with the arrival process in reverse time (in steady state), the stationary departure process is a Poisson process, just like the arrival process. Hence, in steady state, the departure process, which we denote by $D_1(t)$, is a Poisson process with the departure rate equal to the arrival rate. Since the arrival rate is 4 per minute, so is the departure rate. Hence, the distribution is Poisson with mean $\lambda t = 4 \times 2 = 8$. In particular,

$$P(D_1(2) = 8) = \frac{e^{-\lambda t}(\lambda t)^8}{8!} = \frac{e^{-8}8^8}{8!}$$

(i) For what values of p is the second station stable (the number of passengers at the second station does *not* grow without bound as $t \to \infty$), so that the number of passengers waiting at the second station has a proper (finite) steady-state distribution? **(3 points)**

We require that the rate in to station 2 be less than the maximum rate out. The rate in is $\lambda p = 4p$ per minute. The rate out is 0.1 per minute. Hence, we require that

$$4p < 0.1 \quad \text{or} \quad p < 0.025$$

The case of equality is a bit complicated to analyze, but it is bad too. We need strict inequality. To see why, look at the mean in part (b) above. Note that it is infinite if $\rho_1 = 1$.

(j) Suppose that $p = 0.02$. What is the probability that exactly 2 passengers complete the second-stage inspection during a 60 minute interval in steady state? **(3 points)**

We can repeat the reasoning in part (h) at the second of these two queues in series. Let $D_2(t)$ be the number of departures from the second station in the time interval $[0, t]$ in steady state. Since this model is an Markovian open queueing network (OQN) or since the arrival process at the second station is Poisson, the process $D_2(t)$ is also a Poisson process. Since $p = 0.02$, the arrival rate and departure rate at the second station are both $\lambda p = 4 \times 0.02 = 0.08$ per minute. Thus the mean number of departures is $0.08 \times 60 = 4.8$ and

$$P(D(60) = 2) = \frac{e^{-4.8}(4.8)^2}{2!}$$

(k) Again suppose that $p = 0.02$. What is the steady-state probability that simultaneously, at some fixed time, there are 3 passengers at the first station, either being inspected or waiting to be inspected, and 4 passengers at the second station, either being inspected or waiting to be inspected? Justify your answer. **(3 points)**

Let $Q_i(t)$ be the number at station i. We have the product form steady state distribution for two queues in series:

$$\begin{aligned} P(Q_1(\infty) = 3, Q_2(\infty) = 4) &= P(Q_1(\infty) = 3)P(Q_2(\infty) = 4) \\ &= (1 - \rho_1)\rho_1^3 \times (1 - \rho_2)\rho_2^4 \\ &= (1/3)(2/3)^3 \times (0.2)(0.8)^4 \end{aligned}$$

where $\rho_1 = 2/3$ by part (a) and $\rho_2 = (4 \times 0.02)/0.1 = 0.08/0.10 = 0.8$
