

IEOR 3106: Introduction to Operations Research: Stochastic Models

Professor Whitt

Solutions to Homework Assignment 10

Chapter 7: More Renewal Theory

In Ross, read Examples 7.15 and 7.16 in Section 7.4. Read Sections 7.5-7.7, skipping Examples 7.19 and 7.24. (The total required reading is approximately 12 pages.)

Key Concepts in this Assignment:

renewal reward process (Section 7.4)
age of a renewal process (Examples 7.15 and 7.22)
excess of a renewal process (Examples 7.16 and 7.23)
regenerative process (Section 7.5)
alternating renewal process (Section 7.5.1)
semi-Markov process (Section 7.6)
the inspection paradox (Section 7.7)

Do the following exercises at the end of Chapter 7.

26. Hint: Look at Section 7.4.

This is a renewal-reward process problem. The long-run average cost is the expected cost per cycle divided by the expected length of a cycle. A cycle is the interval between successive arrivals of the train. Since customer arrivals occur according to a Poisson process, successive cycles are IID.

The expected length of the cycle is $\frac{N}{\lambda} + K$. The interarrival time between successive arrivals is exponential with mean $1/\lambda$. Thus the time until N arrivals occurs has mean N/λ .

The expected cost per cycle is c times the sum of the numbers of customers times the expected time per cycle with that number of customers. Until the N^{th} customer arrives, there are i customers present for an expected duration of $1/\lambda$. After the N^{th} customer arrives the expected number of customers in the system is $N + \lambda t$ at time t , $0 \leq t \leq K$. The integral from 0 to K is $NK + \lambda K^2/2$. Thus the expected cost per cycle is

$$\begin{aligned} E[\text{cost per cycle}] &= c\left[\frac{0}{\lambda} + \frac{1}{\lambda} + \cdots + \frac{N-1}{\lambda} + NK + \lambda K^2/2\right] \\ &= c\left[\frac{(N-1)N}{2\lambda} + NK + \lambda K^2/2\right]. \end{aligned}$$

Hence the long-run expected cost is

$$\frac{c\left[\frac{(N-1)N}{2\lambda} + NK + \lambda K^2/2\right]}{\frac{N}{\lambda} + K}.$$

$$\begin{aligned}
P(Y(t) > x | A(t) = s) &= P(0 \text{ renewals in the interval } (t, t+x] | A(t) = s) \\
&= P(\text{interarrival time} > x+s | A(t) = s) \\
&= P(\text{interarrival time} > x+s | \text{interarrival time} > s) \\
&= P(X > x+s) / P(X > s) = (1 - F(x+s)) / (1 - F(s)) .
\end{aligned}$$

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Say that the system is “off” at t if the interval crossing t is less than c . Hence the system is off throughout a renewal interval if the total interval length is less than c . Now apply the renewal-reward theorem, to exhibit the long-run proportion as the expected reward per cycle divided by the expected length of the cycle. Hence the long-run proportion of time that $X_{N(t)+1} < c$ is $E[X 1_{\{X \leq c\}}] / E[X]$, where 1_A is the indicator function of the set A ; i.e., $1_A = 1$ on the set A and equals 0 otherwise. Hence, the answer can be rewritten as

$$\frac{E[X 1_{\{X \leq c\}}]}{E[X]} = \frac{\int_0^c t f(t) dt}{E[X]}$$

38. Hint: Look at Section 7.4.

This again is a renewal-reward process problem.

(a) The proportion of his driving time spent driving from A to B is

$$\frac{E[T_{A,B}]}{E[T_{A,B}] + E[T_{B,A}]},$$

where $E[T_{A,B}]$ is the expected time to drive from A to B , while $E[T_{B,A}]$ is the expected time to drive from B to A .

To find $E[T_{A,B}]$ and $E[T_{B,A}]$, we use the elementary formula $d = rt$ (distance = rate $\times$ time). Let S be the driver’s random speed driving from A to B . Then

$$\begin{aligned}
E[T_{A,B}] &= \frac{1}{20} \int_{40}^{60} E[T_{A,B} | S = s] ds \\
&= \frac{1}{20} \int_{40}^{60} \frac{d}{s} ds \\
&= \frac{d}{20} (\ln(60) - \ln(40)) \\
&= \frac{d}{20} (\ln(3/2)) .
\end{aligned}$$

Similarly,

$$E[T_{B,A}] = \frac{1}{2} E[T_{B,A} | S = 40] + \frac{1}{2} E[T_{B,A} | S = 60]$$

$$\begin{aligned}
&= \frac{1}{2} \left(\frac{d}{40} + \frac{d}{60} \right) \\
&= \frac{d}{48}
\end{aligned}$$

(b) Assume that a reward is earned at rate 1 per unit time whenever he is driving at a rate of 40 miles per hour, we can again apply the renewal reward approach. If p is the long-run proportion of time he is driving 40 miles per hour,

$$p = \frac{(1/2)d/40}{E[T_{A,B}] + E[T_{B,A}]} = \frac{1/80}{\frac{1}{20} \ln(3/2) + 1/48} .$$

41. Hint: Look at Example 7.22 and the equilibrium distribution F_e , defined after Example 7.23 (also see the next exercise).

Note that we want the proportion of *time* the age of the machine in use is less than 1 year, not the proportion of machines that have lifetime less than 1 year. Hence we should use the equilibrium distribution of F , denoted by F_e , defined in the next exercise.

(a) We want

$$F_e(1) = \int_0^1 \frac{1 - F(x)}{\mu} dx = \int_0^1 \frac{2 - x}{2} dx = \frac{3}{4} .$$

(b) We want

$$F_e(1) = \int_0^1 e^{-x} dx = 1 - e^{-1} .$$

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Note that we want the proportion of *time* the age of the machine in use is less than 1 year, not the proportion of machines that have lifetime less than 1 year. Hence we should use the equilibrium distribution of F , denoted by F_e , defined in the next exercise.

(a) Note that

$$F_e(x) = \int_0^x \frac{e^{-y/\mu}}{\mu} dy = 1 - e^{-x/\mu} .$$

(b) Note that

$$F_e(x) = \int_0^x \frac{1}{c} dy = \frac{x}{c}, \quad 0 \leq x \leq c .$$

(c) You will receive a ticket if, starting when you park, an official appears within one hour (because then and only then will he appear again the second time before you return in 3 hours).

The time that the official appears has the distribution F_e , which is uniform on the interval $[0, 2]$. Hence, $F_e(1) = 1/2$.

51. Hint: Look at Section 7.7.

This is an example of the inspection paradox. The sampling at departure is a sample from the successive tourists, while the sample at the hotel is a sample at a random time. Since guests staying a longer time are more likely to be sampled at the hotel, it follows that the average time obtained by sampling at the hotel should be longer than the average time found by sampling departures as they are leaving the country. The fact that the averages were 17.8 and 9.0 is consistent with the length of stay being exponential with mean 9.0 days. (The data do not prove that the distribution must be exponential, however.)
