IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2013, Professor Whitt

Numerical Part of Homework Assignment 3: Tuesday, September 17

Markov Chains

Due on Tuesday, October 1, an extra week is given, To be discussed at the recitation on Sunday, September 29.

Computational Exercises with Matlab

This part of the homework assignment emphasizes computational exercises that should be done on a computer. The intended approach is to use MATLAB, which is available to students via the IEOR Computer Lab in Mudd 301. The MATLAB program is also available on the computers in Butler. But you do not need to use MATLAB. You could use MATHEMATICA or some other software.

Here are more details on the computer lab in 301 Mudd. These may change from time to time:

1. Students swipe their cards to get into the room. At this time, there are no time limits on lab access (24x7). Students should not eat or drink in the lab or move any of the furniture or equipment. Wireless connectivity is available for brought-in laptops. Power outlets are available for laptops at the first table in from the front door.

2. Students use their Columbia UNI as their user name and previous password (if they had a previous IEOR account). Students with a brand new IEOR lab account should use the last 4 digits of their social security number (or the last 4 digits of the temporary number assigned instead of the social security number) as their password. After login, the password can be changed by pressing CTRL+ALT+DELETE.

3. To access matlab, find matlab from the list after clicking on start and then programs.

4. No IEOR printers are provided. Users can print to Columbia AcIS printers which are located in the Gussman Lab room 251. There are printer icons for mudd251a mudd251b mudd251c and mudd251d. If students do not have a print quota or their print quota has expired, they can purchase pages at Philosophy Hall, Room 102.

5. The help contact on computer access is Administrator Michael Mostow, mm1812@columbia.edu

Sample MATLAB sessions and programs are posted on the course web page. You can download MATLAB programs and data from the course web page under the Computational Tools link. You should store the MATLAB programs as .m files in your MATLAB work space. You should store the MATLAB data files as .dat files in your MATLAB workspace. You then need to load the data files using a command such as: *load mouse.dat* Then the data will be in the file *mouse* in your workspace. You directly run the MATLAB programs. For example, you enter *absorbing(Q,R)* to run the program *absorbing* obtained from the file *absorbing.m* on the two arguments (matrices) Q and R. See sample sessions on the web page.

MATLAB tutorials can be found online using a search engine such as GOOGLE.

1. Markov Mouse

Refer to the nine-state model of Markov Mouse moving around within the closed maze presented in class and reviewed in the Class Lecture Notes for that day. Or, alternatively, see the computational tools page.

(a) Exploit the structure of the transition matrix (without doing elaborate computation) to deduce what the following transition probabilities must be (exactly or approximately):

$$\begin{array}{cccc} P^{3}_{1,1} & P^{17}_{1,1} & P^{17}_{4,4} & P^{18}_{1,2} \\ P^{21}_{1,2} & P^{21}_{1,6} & P^{21}_{3,4} & P^{22}_{1,1} \\ P^{22}_{1,9} & P^{22}_{9,1} & P^{22}_{1,5} & P^{22}_{5,5} \end{array}$$

(b) Using MATLAB or another computer program, calculate the following matrices:

$$P^{14} \quad P^{16} \quad P^{17} \quad (P^{16} + P^{17})/2$$

(You can turn in the computer printout or a few sample entries.)

(c) Using one of the MATLAB programs **stat.m** or **stationary.m** (which can be downloaded from the course web page), or your own program, calculate the stationary probability vector of P. (See the Class Lecture Notes.) Compare your answer to the calculations in part (b).

(d) What is the long-run proportion of time (steps) that the mouse spends in Room 5?pp

(e) Is there a unique stationary probability vector? If so, what is the stationary probability that the mouse spends in Room 5?

(f) What is the limit (as $n \to \infty$) of the probability that the mouse is in Room 5 at step n, given that the mouse starts in Room 1?

(g) What is the expected number of transitions between successive visits to Room 5?

2. Escaping Markov Mouse

Now allow Markov mouse to escape from the nine-room maze, as discussed in class. In this example we allow escape from rooms 3, 7 and 9 through doors to the outside. Once the mouse escapes, it never returns. Let the probability of leaving each room through any one of the available doors be equally likely. Thus the probability of escaping from each of the rooms 3, 7 and 9, after a visit to that room, is 1/3 in each case. Suppose that we want to keep track of the room from which the mouse eventually leaves the maze; i.e., exiting from Room 3 is to be distinguished from exiting from Room 7 or Room 9.

(a) What is the overall probability transition matrix to model the movement of the mouse now?

(b) Using the MATLAB program **absorbing.m** (which can be downloaded from the course web page), or your own program, calculate the three matrices N, m and B describing the behavior of this absorbing Markov chain. (See the class lecture notes.)

(c) Starting in room 1, what is the probability of the mouse eventually leaving the maze from room 7? Starting in room 1, what is the probability of the mouse eventually leaving the maze from room 9? Starting in room 9, what is the probability of the mouse eventually leaving the maze from room 3?

(d) Starting in room 1, what is the expected number of transitions before the mouse leaves the maze?

(e) Starting in room 2, what is the expected number of visits to room 4 before the mouse leaves the maze?

3. The Canonical Form for a Markov Transition Matrix

The canonical form of a Markov chain transition matrix has the states ordered so that the states in the same communicating class appear together. Moreover, the ergodic (closed communicating) classes appear first, with the transient states appearing below.

The canonical form looks like:

$$P = \begin{pmatrix} P_1 & & & \\ & P_2 & & \\ & & P_3 & \\ R_1 & R_2 & R_3 & Q \end{pmatrix}$$

where the entries of this matrix P are submatrices, with Q and P_i for all i being square matrices. The empty spaces are understood to be matrices of 0's. If Q is $m \times m$ and P_i is $k \times k$, then R_i is $m \times k$. The matrices P_i are understood to be irreducible Markov transition matrices. The states associated with the submatrix Q are the transient states. From any state associated with the submatrix Q, the chain will visit that state only finitely often.

Find the canonical form of each of the following Markov transition matrices:

(a)

$$P = \begin{pmatrix} 0.1 & 0.0 & 0.0 & 0.9 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.0 & 0.6 \\ 0.3 & 0.3 & 0.0 & 0.4 & 0.0 \\ 0.3 & 0.0 & 0.0 & 0.7 & 0.0 \\ 0.0 & 0.7 & 0.0 & 0.0 & 0.3 \end{pmatrix}$$

(b)

$$P = \begin{pmatrix} 0.0 & 0.2 & 0.0 & 0.0 & 0.8 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \end{pmatrix}$$

(c)

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.1 & 0.0 & 0.2 & 0.2 & 0.2 & 0.0 \\ 0.0 & 0.2 & 0.0 & 0.0 & 0.0 & 0.8 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.1 & 0.0 & 0.0 & 0.0 & 0.9 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.1 & 0.6 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.7 & 0.0 & 0.0 & 0.0 & 0.3 & 0.0 \\ 0.1 & 0.0 & 0.0 & 0.4 & 0.0 & 0.0 & 0.0 & 0.5 \end{pmatrix}$$

Compute high powers of each transition matrix, after it has been put in canonical form. Observe that the canonical form is useful to reveal the key structure.

4. More on Stationary Probability Vectors

Calculate the stationary vector of the following Markov transition matrix. Compare to high powers of the matrix P.

$$P = \begin{pmatrix} 0 & 0 & .25 & .75 & 0 & 0 \\ 0 & 0 & .5 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & .875 & .125 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$