IEOR 3106: Introduction to Operations Research: Stochastic Models

Professor Whitt

Solutions to Homework Assignment 4

Even More Markov Chains

Read Sections 4.5.1, 4.5.2, 4.6 and part of 4.8. In Section 4.8 read pages 249-255, up to Example 4.37. (In the 9th edition, the covered material in Section 4.8 is on pages 236-242, up to Example 4.33.

Do the following exercises at the end of Chapter 4. Turn in all except for ones with answers in back:

41. (Answer in back)

42.

This exercise is similar to the previous one.

(a) This first double sum is the long-run proportion of transitions that go from a state in the set A to a state in the complement of A (the subset of remaining states), denoted by A^c.

(b) This second double sum is the long-run proportion of transitions that go from a state in the set A^c to a state in the set A.

(c) Between any two transitions from A to A^c, there must be a transition from A^c to A. Similarly, between any two transitions from A^c to A, there must be a transition from A to A^c. Hence the long-run proportion of transitions from A to A^c must equal the long-run proportion of transitions from A^c to A.

46.

(i) The key is to define a good state. Let the state be the number of umbrellas at his present location. The possible values for the states are $0, 1, \ldots, r$, so there are $r + 1$ states.
With this definition for the state, the transition probabilities are

$$P_{0,r} = 1 \quad \text{(He can take no umbrella, but finds all at his destination.)}$$

If it does not rain, he takes no umbrella. Since the probability of not raining is $1 - p$, we have

$$P_{i,r-i} = 1-p \quad \text{for} \quad 1 \le i \le r . \quad \text{(He takes no umbrella and find the others at the destination.)}$$

Otherwise it rains, and he takes an umbrella with him, so there is 1 more umbrella at his destination, yielding:

$$P_{i,r-i+1} = p \quad \text{for} \quad 1 \le i \le r .$$

(ii) It suffices to show that $\pi = \pi P$. Consequently, it suffices to show that

$$\pi_j = \sum_{i=0}^{r} \pi_i P_{i,j}, \quad 0 \le j \le r .$$

These equations become

$$\begin{aligned}
\pi_r &= \pi_0 + \pi_1 p \\
\pi_j &= \pi_{r-j}(1-p) + \pi_{r-j+1} p, \quad 1 \le j \le r-1, \\
\pi_0 &= \pi_r(1-p)
\end{aligned}$$

Start with $\pi_r = \frac{1}{r+1-p}$:

$$\pi_r = \frac{1}{r+1-p} = \frac{1-p}{r+1-p} + \frac{1}{r+1-p}p ,$$

as desired.

Similarly, the other equations hold.

He gets wet if there is no umbrella and it rains. The steady-state limiting probability of that event is

$$\pi_0 p = \frac{p(1-p)}{r+1-p} .$$

(iv) Differentiate the probability obtained in part (iii) for $r = 3$:

$$\frac{d}{dp} \frac{p(1-p)}{4-p} = \frac{p^2 - 8p + 4}{(4-p)^2}$$

Setting the derivative equal to 0, we get

$$p = \frac{8 - \sqrt{48}}{2} \approx 0.55 .$$

You can check that the extreme value yields a mximum, not a minimum.

47.

This exercise is somewhat similar to the expansion of the states to make a non-Markov process Markov, which we considered in Example 4.4 and Exercise 2.

Here, however, the original process is already Markov. The answer is in the back of the book.

49.

(i) No, at time n, $n \ge 2$, the future conditional on the past and present depends on the outcome of the coin flip at time 1.

The limiting probability can still be computed. It is

$$\lim_{n\to\infty} P(X_n = i) = p\pi_i^1 + (1-p)\pi_i^2 .$$

(ii) Yes, the successive states now are a Markov chain. The transition probabilities now are:

$$P_{i,j} = pP_{i,j}^{(1)} + (1-p)P_{i,j}^{(2)} \ .$$

56.

Recognize that this is the gambler's ruin problem discussed in Section 4.5.1. By formula (4.14) in the text (formula (4.13) in the 7th edition), The probability of winning is the probability of reaching level $n + m$ before hitting 0, starting in level m. That probability is

$$P(win) = \frac{1 - (q/p)^m}{1 - (q/p)^{n+m}} \ ,$$

where $q \equiv 1 - p$.

57.

Let A be the event that all steps are visited by time T, the time of first return to state 0. Conditioning on the direction of the first step, we can transform the problem into two applications of the gambler's ruin problem, which was just discussed in the previous exercise. Specifically, we get

$$
\begin{aligned}
P(A) &= P(A|\text{go clockwise at first})p + P(A|\text{go counterclockwise at first})(1-p) \\
&= p\frac{1 - (q/p)}{1 - (q/p)^n} + (1-p)\frac{1 - (p/q)}{1 - (p/q)^n}
\end{aligned}
$$

These probabilities are the probabilities that a gambler starting with 1 will reach n before going broke. The two directions change the role of p and q.

76.

Use the formula for the steady-state probability distribution given at the end of Example 4.32, namely,

$$\pi_i = \frac{\sum_j w_{i,j}}{\sum_i \sum_j w_{i,j}} \ ,$$

where $w_{i,j}$ is the weight on arc (i, j), which here we take to be 1 if the knight can move from square i to square j. There are 2 possible moves from each corner. There are 4 squares from which there are 2 possible moves (the corners). There are 8 squares from which there are 3 possible moves. There are 20 squares from which there are 4 possible moves. There are 16 squares from which there are 6 possible moves. There are 16 squares from which there are 8 possible moves. Thus, the sum of the weights is 336. Let state 1 be a corner. Then the steady-state probability of being at a corner is $\pi_1 = 2/336 = 1/168$. Hence the expected number of steps for the Markov knight to return to the corner from which it started is 168. At this last step we use Remark (ii) in Section 4.4, which states that

$$\pi_i = \frac{1}{m_{i,i}} \ .$$

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where $m_{i,i}$ is the mean time to return to state i.