IEOR 3106: Introduction to Operations Research: Stochastic Models

Solutions to Homework Assignment 8.

Chapter 6: More Continuous-Time Markov Chains

Read Sections 6.6-6.8 in Ross. Do the following exercises at the end of Chapter 6.

14. Just like Problem 13 in the last assignment, this is an $M/M/1/1$ queueing model, having one server and one extra waiting space (a total capacity of 2).

Let $N(t)$ be the number of cars at the gas station at time t. The process $\{N(t) : t \geq 0\}$ is a birth-and-death process with state space $\{0, 1, 2\}$.

The birth rates (per hour) are

$$\lambda_0 = 20 \quad \text{and} \quad \lambda_1 = 20 \ .$$

There is no (zero) birth rate in state 2. The death rates are

$$\mu_1 = 12 \quad \text{and} \quad \mu_2 = 12 \ .$$

There is no (zero) death rate in state 0.

The steady-state distribution can be found directly quite easily by solving local balance equations as indicated in Remark (iii) in Section 6.5. There is a standard form for any BD process (see (6.20), which we exploit below.

The steady-state vector α has the form

$$\alpha_i = \frac{r_i}{r_0 + r_1 + r_2} \quad \text{for} \quad 0 \leq i \leq 2 \ ,$$

where

$$r_0 = 1, \quad r_1 = \frac{\lambda_0}{\mu_1} \quad \text{and} \quad r_2 = r_1 \times \frac{\lambda_1}{\mu_2} \ ,$$

i.e., $r_0 = 1$, $r_1 = 20/12 = 5/3$ and $r_2 = 25/9$, so that the specific steady-state probabilities are

$$\alpha_0 = \frac{9}{49}, \quad \alpha_1 = \frac{15}{49}, \qquad \text{and} \quad \alpha_2 = \frac{25}{49} \ .$$

(a) The fraction of time that the system is busy is $\alpha_1 + \alpha_2 = 40/49$. That is the proportion of time that the attendant spends servicing cars.

(b) The proportion of customers that do not enter the shop coincides with the proportion of time that the system is full (because the arrivals occur at a constant rate over all time). The proportion of time that the system is full is $\alpha_2 = 25/49$.

20. Let the state be the number of machines that are down. Then the state space is the set $\{0, 1, 2\}$. The stochastic process $\{X(t) : t \geq 0\}$, where $X(t)$ is the number of machines down at time t, is a birth-and-death process, because the process is a CTMC that can move only to a neighboring state at each transition.

The birth rates are $\lambda_i = \lambda$ for $i = 0, 1$; the death rates are $\mu_i = \mu$ for $i = 1, 2$. This model coincides with a $M/M/1/1$ queueing model, which has 1 server and 1 extra waiting space. Customer arrivals in the queue correspond to failures in the machines. Service times in the queue correspond to repair times for the machines.

(a) Referring to pages 356-359, we see that we want $E[T_0 + T_1]$, because T_0 is the time to go from state 0 to state 1, while T_1 is the time to go from state 1 to state 2. Then, following the top of page 357, we get

$$E[T_0 + T_1] = E[T_0] + E[T_1] = (\frac{1}{\lambda}) + (\frac{1}{\lambda} + \frac{\mu}{\lambda^2}) = \frac{2}{\lambda} + \frac{\mu}{\lambda^2} \ .$$

(b) To treat the variances, we look at the end of Section 6.3. In particular,

$$Var(T_0 + T_1) = Var(T_0) + Var(T_1) = (\frac{1}{\lambda^2}) + (\frac{1}{\lambda(\lambda+\mu)} + \frac{\mu}{\lambda^3} + \frac{\mu}{\mu+\lambda}(\frac{2}{\lambda} + \frac{\mu}{\lambda^2})^2 \ .$$

The steady-state distribution has a simple form for a birth-and-death process. Let α_i be the steady-state probability of having i failed machines. Here, we want

$$\alpha_0 + \alpha_1 = 1 - \alpha_2 = \frac{1 + \lambda/\mu}{1 + \lambda/\mu + (\lambda/\mu)^2} \ .$$

23. Let the state be the number of machines that are down. The state space is thus $\{0, 1, 2, 3\}$. The number of machines that are down at time t is a BD process with birth rates $\lambda_0 = 3/10$, $\lambda_1 = 2/10$, $\lambda_2 = 1/10$ and death rates $\mu_1 = 1/8$, $\mu_2 = 2/8$, $\mu_3 = 2/8$.

The steady-state vector α has the form

$$\alpha_i = \frac{r_i}{r_0 + r_1 + r_2 + r_3} \quad \text{for} \quad 0 \le i \le 3 \ ,$$

where

$$r_0 = 1, \quad r_1 = \frac{\lambda_0}{\mu_1}, \quad r_2 = r_1 \times \frac{\lambda_1}{\mu_2}, \quad r_3 = r_2 \times \frac{\lambda_2}{\mu_3} \ ,$$

i.e., $r_0 = 1$, $r_1 = 24/10 = 12/5$, $r_2 = 48/25$ and $r_3 = 96/125$, so that

$$\alpha_0 = \frac{125}{761}, \quad \alpha_1 = \frac{300}{761}, \quad \alpha_2 = \frac{240}{761}, \qquad \text{and} \quad \alpha_3 = \frac{96}{761} \ .$$

(a) The average number of machines not in use is

$$(0 \times \alpha_0) + (1 \times \alpha_1) + (2 \times \alpha_2) + (3 \times \alpha_3) = \frac{1068}{761} \ .$$

(b) The proportion of time both repairman are busy is

$$\alpha_3 + \alpha_2 = \frac{336}{761} \ .$$

24. The answer is in the back of the book. Note that this is an $M/M/1/\infty$ queueing model, but with an unusual interpretation. Here we want to let the arriving "customers" in

the queueing model be the arriving taxis. A service occurs when a customer arrives and takes one of the taxis away. Thus the arrival rate is $\lambda = 1$ and the service rate is $\mu = 2$. This is a BD process with birth rate in each state equal to the arrival rate, and death rate in each state (except state 0) equal to the service rate.

26. We want to deduce, possibly counter to intuition, that the number of customers present in an $M/M/s$ queue in steady state is independent of the sequence of past departures. Toward that end, we exploit time reversibility in Section 6.6. The reverse-time system is stochastically equivalent to the standard forward-time system. Past departures in the standard forward-time system correspond to future arrivals in the reverse time system. Since the arrival process is a Poisson process, the departure process also must be a Poisson process. In the reverse time system, the Markov property (applied to the Poisson arrival process) makes the number in the system at any time independent of the future arrival process. So, in the forward-time system the number in system is necessarily independent of past departures.

27. This question is related to Proposition 6.5 and Corollary 6.6. In the $M/M/s$ queue if you allow the service rate to depend on the number in the system but in such a way so that it is ergodic, the process $\{X_t, t \geq 0\}$ of the number of customers in the system is still an ergodic birth-death process. The difference is that the death rate of the process will be a function of the process itself. So by Corollary 6.6, as long as the traffic intensity is kept less than one so that a steady state exists for this queue system, the output process of customers is a Poisson process with the same as the arrival process.

If the service rate μ remains unchanged but $\lambda > s\mu$, the condition in Corollary 6.6 is violated, so one can no longer conclude that the output process of customers is Poisson with rate λ. In fact, under the condition $\lambda > s\mu$, after the process has been in operation for a long time, the process $\{X_t, t \geq 0\}$ of the number of customers in the system will blow up, but the output process is actually a Poisson process with rate $s\mu$ since all servers will be constantly utilized.

28. The answer is in the book. The focus, of course, is on time reversibility, which is discussed in Section 6.6. If α denotes the steady-state probability vector and Q denotes the rate matrix, then time reversibility holds when the local-balance equations are satisfied, i.e., when

$$\alpha_i Q_{i,j} = \alpha_j Q_{j,i} \quad \text{for all} \quad j \neq i \ .$$

30. Let $\{X_t, t \geq 0\}$ be the process of the location of the particle on the graph. It is evident that this process is an ergodic Markov chain. Here we consider the process being in steady state. If the process is in state i, i.e., the particle is at node i, then it will make a transition to node $j (\neq i)$ along the arc (i, j) with rate $Q_{ij} = \lambda_{ij}$. Similarly if the process is in state j, it will make a transition to node $i (\neq j)$ along the arc (j, i) (same as (i,j)) with rate $Q_{ji} = \lambda_{ji} = \lambda_{ij}$. Consider the reverse time process, if the reverse process is in state i, then it will make a transition to node $j (\neq i)$ along the arc (i, j) in backward time with rate $Q^*_{ij} = \lambda_{ij}$. So for all i, j, $Q^*_{ij} = Q_{ij}$, which implies that this CTMC is time reversible. Then by $P_i Q_{ij} = P_j Q_{ji}$ where P_i is the proportion of time that the particle is at node j, we have $P_i = P_j$ for all i, j. Therefore, $P_j = 1/n$ for all j.

33. Again, the answer is in the back of the book.