IEOR 3106: Introduction to Operations Research: Stochastic Models

Professor Whitt

Solutions to Homework Assignment 9

Chapter 7: Renewal Theory and its Applications

In Ross, read Sections 7.1-7.3 up to (not including) Example 7.8. Skip Remark (ii) in Section 7.2 and Examples 7.1 and 7.3. Also Read Section 7.4 up to (not including) Example 7.13. (The total required reading is approximately 12 pages.)

Do the following exercises at the end of Chapter 7.

1. Hint: See the beginning of Section 7.2.

The defining relation is (7.1). The property that does hold is (7.2). This example illustrates that correctness depends on whether or not the inequalities are strict or not.

The answers are (a) yes, (b) no, and (c) no.

It is easy to see that (a) is equivalent to (7.2), while the others are not. It is not difficult to give concrete examples.

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2. Hint: See Sections 2.2.4 and 7.2. Recall that the sum of independent Poisson random variables has a Poisson distribution.

Since X_n is Poisson with mean μ, S_n is Poisson with mean $n\mu$. From (7.3),

$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n+1) \\ &= P(S_n \leq t) - P(S_{n+1} \leq t) \\ &= \sum_{k=0}^{k=\lfloor t \rfloor} \frac{e^{-n\mu}(n\mu)^k}{k!} - \sum_{k=0}^{k=\lfloor t \rfloor} \frac{e^{-(n+1)\mu}((n+1)\mu)^k}{k!} , \end{aligned}$$

where $\lfloor t \rfloor$ is the greatest integer less than or equal to t.

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3. Hint: See Example 7.2 and following Remark (i).

The answer is in the back of the book.

The one-to-one relationship is easy to see when we use Laplace transforms. For a nonnegative random variable X, its Laplace transform is $E[e^{-sX}]$, where s is a new variable (in general regarded as a complex variable, but that is not crucial here). If X has probability density function (pdf) f, then the Laplace transform of X coincides with the Laplace transform of the pdf f, denoted by $\hat{f}(s)$; i.e.,

$$\hat{f}(s) = E[e^{-sX}] = \int_0^\infty e^{-sx} f(x)\, dx \ .$$

It is known that there is a one-to-one correspondence betweeen a pdf and its Laplace transform. Moreover, it turns out that the Laplace transform of the cdf F, where

$$F(t) = \int_0^t f(u)\, du, \quad t \geq 0 \ ,$$

is $\hat{f}(s)/s$. In addition, if S_n is the sum of n independent and identically distributed (IID) random variables, each with pdf f, then

$$E[e^{-sS_n}] = E[e^{-sX_1}]^n = \hat{f}(s)^n.$$

Since the renewal function $m(t)$ satisfies (see Section 7.2)

$$m(t) = \sum_{n=1}^{\infty} P(S_n \leq t) \ ,$$

the Laplace transform of $m(t)$, defined by

$$\hat{m}(s) = \int_0^\infty e^{-st} m(t)\, dt \ ,$$

satisfies the relation

$$\hat{m}(s) = \sum_{n=1}^{\infty} \hat{f}(s)^n / s = \frac{\hat{f}(s)}{s(1 - \hat{f}(s))} \ .$$

We need the s in the denominator because we have the cdf of S_n, not its density, when we look at $P(S_n \leq t)$; i.e., the Laplace transform of $P(S_n \leq t)$ is $\hat{f}(s)^n/s$.

From the last equation, we see that we can solve for $\hat{f}(s)$ given $\hat{m}(s)$ by

$$\hat{f}(s) = \frac{s\hat{m}(s)}{1 + s\hat{m}(s)} \ .$$

Hence we do indeed have the one-to-one reflationship between the pdf of X_n, denoted by f, and the renewal function m, as claimed.

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4. Hint: Consider the special case of deterministic times between renewals.

In general the answers to (a)-(c) are all NO. However, the answers are all YES for the special case of a Poisson process. It turns out that the answers are NEVER yes for renewal processes if both processes are not Poisson, but that is somewhat hard to prove.

To construct a specific example, let one process be a Poisson process with rate 1, and let the other renewal process have constant times between renewals, with value 1 for all n.

(a) No. If the first interarrival time in $N(t)$ is $1/4$, then the second is sure to be less than $3/4$. Hence the interarrival times in $N(t)$ are not independent.

(b) No. The probability that the first interarrival time is 1 is e^{-1}. The probability that the second interarrival time is 1 is necessarily different. The second interarrival time can be exactly 1 only if no Poisson arrivals have occurred by time 2. That probability is e^{-2}.

(c) No, because of parts (a) and (b).

It is important that all three of these properties do hold when the two component processes N_1 and N_2 are independent Poisson processes. Then the superposition process N is itself a Poisson process (a very special case of a renewal process).

7.

The mean time between successive jobs is $3+2=5$ months. The rate at which Mr. Smith gets new jobs is 1 per every 5 months or 2.4 jobs per year.

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8. Hint: Look at Section 7.3.

The answer is in the book.

21.

Note that this example is an $M/G/1/0$ queue, where the service times are IID with a general distribution. In this model there is a single server, but no extra waiting space. There is either one customer in the system or zero.

Let m be the mean of the general service time. The proportion of time that the server is busy is

$$\frac{m}{m+(1/\lambda)} .$$

We obtain this by using the renewal-reward-process framework (see the middle of page 418). A cycle is a service time plus the following interarrival time. We want to compute the expected reward per cycle divided by the expected length of a cycle. Reward is earned at rate 1 when the server is busy. The expected reward per cycle is the expected service time. Thus we get the formula above.

22. Hint: Look at Section 7.4.

The answer is in the back of the book.

Old 23. I had meant to ask Problem 7.23 in the previous edition:

If H is the uniform distribution over $(0,8)$ and $C_1=4$, $C_2=1$ and $R(T)=4-(T/2)$, then what value of T minimizes Ms. Jones' long-run average cost in Exercise 7.22?

Hint: Look at Section 7.4.

From the previous problem 7.22, we want to minimize

$$\frac{4+\frac{T-2}{6}-[4-\frac{T}{2}][\frac{8-T}{6}]}{\int_2^T x\frac{dx}{6}+T[\frac{8-T}{6}]}$$

over $2 \leq T \leq 8$.

That expression reduces to

$$\frac{18T - 20 - T^2}{16T - 4 - T^2} \ .$$

Using calculus, the quantity can be shown to be increasing in T in the interval $[2, 8]$, so that the optimal value is $T = 2$.

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23. in current addition

I did not mean to ask this question, and I did not mean to cover Wald's equation at this point, but you could look it up in the index and see that it is discussed in Problems 7.13-7.16. Reading that, you would learn something good to know, but again. In fact, this reasoning will come up in our analysis of Brownian motion in Chapter 10.

In case you did try to do this question, we now provide you feedback. Further explanation about what follows appears in the previous problems mentioned above. More will appear in the last three lectures.

Let T be the number of bets made before the gambler stops. When he stops, there are two possible results. Either his fortune reaches N, or he loses all his money. So we have:

$$\sum_{j=1}^{T} X_j = -i \ or \ N - i \ .$$

if $E[T] < \infty$, we can apply the Wald's Equation:

$$E[\sum_{j=1}^{T} X_j] = E[X_1]E[T] \ .$$

Since the event$\{T = n\}$ is independent of the random variables after n, i.er., of X_{n+1}, X_{n+2}, the random variable T is a stopping time; see Problem 7.13. ...

When $p \neq \frac{1}{2}$, $E[T] < \infty$

$$\begin{aligned} E[T] &= \frac{E[\sum_{j=1}^{T} X_j]}{E[X_1]} \\ &= \frac{(N-i)P(wins\ N-i\ when\ stop) + (-i)P(loses\ i\ when\ stop)}{p + (-1)\cdot(1-p)} \\ &= \frac{(N-i)\cdot P_i + (-i)\cdot(1-P_i)}{2p-1} \\ &= \frac{N \cdot P_i - i}{2p-1} \end{aligned}$$

Here $P_i = \frac{1-(\frac{1-p}{p})^i}{1-(\frac{1-p}{p})^N}$. See Section 4.5.1, Gambler's Ruin problem.

When $p = \frac{1}{2}$, it's a little bit more complicated since $2p - 1 = 0$ *and* $NP_i - i = 0$. The formula above doesn't work. However we can do it in the following way:

Let T_i be the number of bets made before he stops, starting from i.

$$\begin{aligned}
E[T_0] &= 0 \\
E[T_i] &= E[T_i|X_1 = 1]\cdot p + E[T_i|X_1 = -1]\cdot(1-p) \\
&= (1 + E[T_{i+1}])\cdot\frac{1}{2} + (1 + E[T_{i-1}])\cdot\frac{1}{2} \\
&= 1 + \frac{1}{2}E[T_{i+1}] + \frac{1}{2}E[T_{i-1}] \qquad i = 1, 2, ..., N-1 \\
E[T_N] &= 0
\end{aligned}$$

Solving this set of equations gives us the values of $E[T_0], E[T_1], ..., E[T_N]$ as:

$$E[T_i] = (N - i)\cdot i\ .$$

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9. Hint: Let T be the time it takes to complete a job. Let W be the time it would take to complete the first job attempted. Let S be the time of the first shock. To compute $E[T]$, develop an equation for it, by conditioning on the possible outcomes of W; i.e., compute $E[T]$ by computing $E[E[T|W]]$. To compute $E[T|W = w]$, compute $E[T|W = w, S = x]$, multiply by the density $f_S(x)$ and integrate over x.

The job completions constitute renewals. Following the hint,

$$E[T|W = w] = \int_0^\infty E[T|W = w, S = x]\lambda e^{-\lambda x}\, dx\ .$$

However,

$$E[T|W = w, S = x] = x + ET \quad \text{if} \quad x < w\ ,$$

while

$$E[T|W = w, S = x] = w \quad \text{if} \quad x \geq w\ .$$

Thus,

$$E[T|W] = (E[T] + 1/\lambda)(1 - e^{-\lambda W}).$$

Taking expectations yields

$$E[T] = (E[T] + 1/\lambda)(1 - E[e^{-\lambda W}])\ ,$$

which in turn implies that

$$E[T] = \frac{(1 - E[e^{-\lambda W}])}{\lambda E[e^{-\lambda W}]}\ .$$

Note the role played by the Laplace transform of W, $\hat{f}_W(s) \equiv E[e^{-sW}]$ for $s = \lambda$.
