IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2013, Professor Whitt

Class Lecture Notes: Tuesday, September 3

Conditional Probability

We discussed several problems in Chapter 1 of Ross. I emphasized several key points:

1. Probability theory is a **branch of mathematics**, so it is important to pay attention to **definitions** and axioms (see the beginning of Section 1.3 in the Ross textbook).

You need to recall some elementary **set theory**. We use braces to denote a set, as appear in the definition of the events S and E in Problem 1.18 below. Just to focus on one pedantic (but useful) detail, note that x, $\{x\}$ and $\{\{x\}\}$ are different objects: x is an element of the set $\{x\}$, while $\{x\}$ is an element of the set $\{\{x\}\}$; x is not an element of the set $\{\{x\}\}$. A set containing the element x is not the same as the element x itself. It may help to Google "set theory." That is, look at Wikipedia, PlanetMath or Wolfram's Mathworld. One or two pages of reading should suffice.

Formally, a **probability measure** assigns probabilities to subsets of the sample space; those subsets are called events. See Sections 1.1-1.3 of Ross.

2. We are focusing on **problem solving**. For that purpose, a good general strategy is **divide and conquer**: break the problem into smaller pieces that are easier to analyze. Skipping steps can cause errors.

3. It is helpful to draw **pictures**.

In particular, Chapter 1 emphasizes that a key idea overall is to remember and apply the definition of **conditional probability**:

$$P(A|B) \equiv \frac{P(AB)}{P(B)} \ ,$$

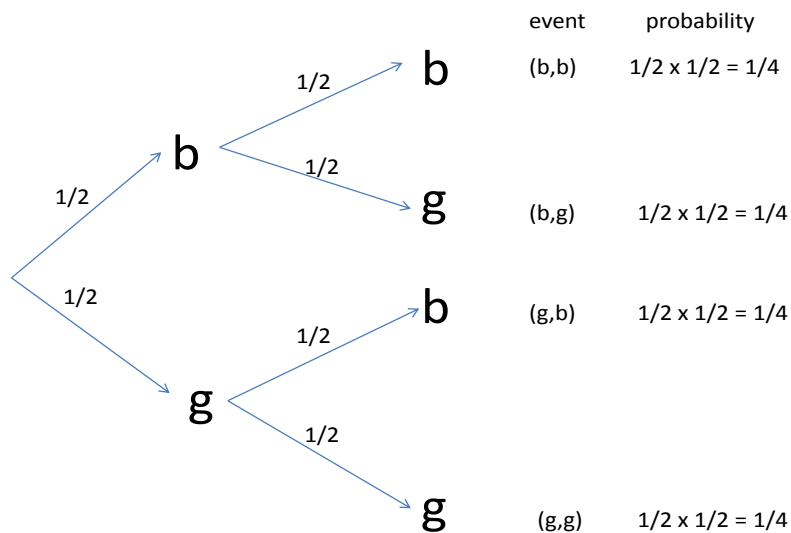where $AB \equiv A \cap B$ denotes the *intersection* of the events (sets) A and B.

The following are exercises at the end of Chapter 1 in Ross.

1.18 (a) A family has two children. What is the probability that both are girls, given that at least one is a girl? (Assume that each child is equally likely to be a boy or a girl.)

(b) Does the answer change if we rephrase the question: What is the probability that both are girls, given that the older child is a girl?

ANSWER: Start by drawing a picture, as shown in the figure on top of the next page. This is a probability tree, showing how the events evolve over time. It is not necessary to draw the picture, but it helps us formalize the problem correctly. The picture shows how we can

A probability tree

		event	probability
1/2 → b	b	(b,b)	1/2 x 1/2 = 1/4
1/2 → g		(b,g)	1/2 x 1/2 = 1/4
1/2 → b	g	(g,b)	1/2 x 1/2 = 1/4
1/2 → g		(g,g)	1/2 x 1/2 = 1/4

correctly put the problem into the mathematical framework. The *sample space* is the *set*

$$S = \{(g,g), (g,b), (b,g), (b,b)\},$$

where (b,g) means that the first (eldest or older) child is a boy and the second (younger) child is a girl. Then events are subsets of the sample space. The event we are concerned with is $B = \{(g,g)\}$, the event that both children are girls.

The key idea here is that the events E and L are different, where E is the event that the eldest child is a girl, while L is the event that at least one child is a girl. the events E and L are:

$$E = \{(g,g), (g,b)\}$$

(eldest girl is first is girl) and

$$L = \{(g,g), (g,b), (b,g)\} \ .$$

Thus $P(B|E) = 1/2$, while $P(B|L) = 1/3$. Use the definition of conditional probability:

$$P(B|E) = \frac{P(BE)}{P(E)} \ .$$

1.28 If the occurrence of event B makes event A more likely, does the occurrence of event A make event B more likely?

For example, suppose the A is the event that the Nasdaq stock index goes up on a given day, while B is the event that the weather is nice on that particular day (defined in some precise way, e.g., temperature remains between 50 degrees and 75 degrees, humidity remains below 30% and the measured precipitation is less than 0.01 inch).

ANSWER: The key idea here is to move from a "gut reaction" toward a reasoned approach. That can be achieved by formulating the problem precisely. There might be some question

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about the formulation, but we separate the issue of the formulation from the reasoning given the formulation. Clarity is often achieved by the formulation. The question can be expressed as follows: If $P(A|B) \geq P(A)$ is it necessarily true that $P(B|A) \geq P(B)$. Once the question has been so formulated, it is easy to see that the answer is indeed "Yes." As a second step, apply the definition of conditional probability to write down what the conditional probabilities are. You are then left with trivial algebra.

1.29 Suppose that $P(E) = 0.6$. What can you say about $P(E|F)$ when
(a) E and F are mutually exclusive?
(b) $E \subseteq F$?
(c) $F \subseteq E$?

ANSWER: It is convenient to draw a picture.

A Venn Diagram
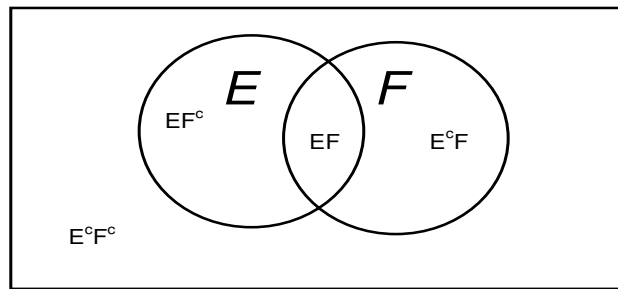
The relevant picture here is a **Venn diagram**, as shown in Figure 1. It shows the two events E and F as subsets within the sample space. When we consider the two events together, these two events determine four events: EF, EF^c, E^cF and E^cF^c. Now turning to the specific questions, recall that "mutually exclusive" means that the intersection is empty; the two sets E and F have no elements in common; i.e., $EF = \phi$. The answers are: (a) $P(E|F) = P(EF)/P(F) = 0$, (b) $P(EF)/P(F) = P(E)/P(F) = 0.6/P(F) \geq 0.6$, (c) $P(EF)/P(F) = P(F)/P(F) = 1.0$.

1.33 In a class there are four freshman boys, six freshman girls, and six sophomore boys. How many sophomore girls must be in the class if gender and class are to be independent when a student is selected at random?

ANSWER: Again we can use a Venn diagram. Let B be the set of boys, with its complement B^c being the set of girls. Let F be the set of freshmen, with its complement F^c being the set of sophomores. Let n be the number of sophomore girls, i.e., the number of elements in the set B^cF^c. Then we have the Venn diagram in the figure below. There is the single unknown n. Make an equation with one unknown and solve it. Let the definition of **independence** give

Venn Diagram:
Boys (B) and Girls; Freshman (F) and Sophomores

B BF^c 6

BF 4

F B^cF 6

B^cF^c

n = ?
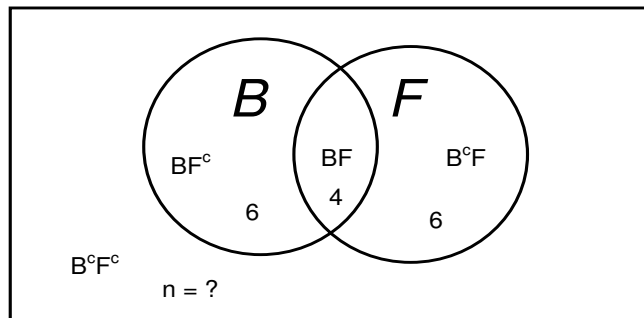
you the equation:

$$P(FB) = P(F)P(B)$$

so that the equation becomes

$$\frac{4}{16+n} = \frac{10}{16+n} \times \frac{10}{16+n} \, ,$$

from which it is easy to see that the answer is $n = 9$. Any of the other events would yield the same answer; e.g., we could have used instead

$$P(SG) = P(S)P(G) \, .$$

1.42 There are three coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. Suppose that one of these three coins is selected at random and flipped. What is the conditional probability that the randomly selected coin was the two-headed coin, given that the outcome of the flipping showed heads?

ANSWER: This is the typical example from Chapter 1. We use **Bayes theorem** to **reverse the conditional probability**: get $P(B|A)$ from $P(A|B)$ (and other information). See examples in Section 1.6. Let T be the event that the two-headed coin is selected; let F be the event that the fair coin is selected; and let B be the event that the biased coin is selected. Here we are given $P(H|T) \equiv P(Head|Two-headed)$ and we want compute $P(T|H)$. This can be done directly algebraically, but it again helps to draw a picture. Again draw a probability tree. From the root, draw three branches to show the possibilities of which coin is chosen; let the probability weight on each branch be 1/3. Then from each of these branches, make further branches showing the outcome of the coin toss, conditional on the selected coin.

probability tree

T = two-headed; F = fair; B = biased; H = head
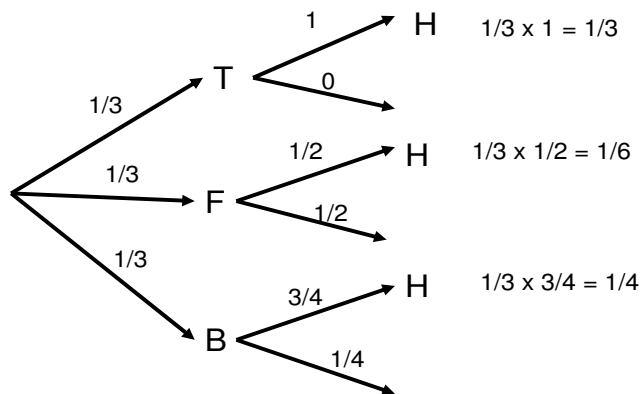
In particular, we have

$$P(T|H) = \frac{P(TH)}{P(H)} = \frac{P(T)P(H|T)}{P(T)P(H|T) + P(F)P(H|F) + P(B)P(H|B)} = \frac{1/3}{1/3 + 1/6 + 1/4} = \frac{4}{9} \ .$$

1.45 An urn contains b black balls and r red balls. One of the balls is drawn at random. We then put that ball back in the urn, but we also put c more balls of the same color as the drawn ball into the urn. Now suppose we draw another ball. Show that the probability that the first ball drawn was black, given that the second ball drawn was red is $b/(b+r+c)$.

ANSWER: This is yet another application of Bayes theorem. Again, draw a probability tree. From the root, draw two branches showing the first ball drawn. Then from each of these outcomes, draw two branches showing the second ball drawn, given the first ball drawn. When inserting the probabilities, be sure to add the c balls of the right color after the first draw.

Now we apply Bayes theorem to calculate the desired conditional probability:

$$P(B1|R2) = \frac{P(B1 \cap R2)}{P(R2)} = \frac{(br/[(b+r)(b+r+c)])}{(br/[(b+r)(b+r+c)]) + (r(r+c)/[(b+r)(b+r+c)])} \ ,$$

so that

$$P(B1|R2) = \frac{br}{br + r(r+c)} = \frac{b}{b+r+c} \ .$$

An urn with black and red balls

Bj = black ball drawn on draw j; Rj = red ball drawn on draw j