IEOR 3106: Introduction to Operations Research: Stochastic Models

Professor Whitt

Reversibility

1. Four Problems

(1) The Knight Errant (The Random Knight)

A knight is placed alone on one of the corner squares of a chessboard (having $8 \times 8 = 64$ squares). What is the expected total number of moves required for the knight to first return to its initial position, if we assume that the knight moves randomly, taking each of its legal moves in each step with equal probability?

(2) A Big Closed Maze for Markov Mouse

Suppose that the closed maze for Markov mouse is enlarged to be 10×20 instead of 3×3; i.e., it now has $10 \times 20 = 400$ rooms instead of $3 \times 3 = 9$ rooms, but still arranged in a rectangular fashion, with doors connecting neighboring rooms. Now there are 10 rows of rooms, with 20 rooms in each row. There are doors connecting neighboring rooms on each row. And there are doors connecting neighboring rooms on each column.

Suppose that, just as before, the mouse moves randomly according to a Markov chain, moving to one of the available neighboring rooms on each move, with each of the available alternatives chosen with equal probability.

Suppose that the mouse starts in Room 1 (in the upper lefthand corner). What is the expected total number of moves required for the mouse to first return to this initial room?

(3) Random Walk on a Finite Graph

See Section 4.8 of Ross.

A finite graph consists of a finite set of **vertices** V (or nodes) plus a set of **arcs**. Some pairs of vertices are connected by arcs and some pairs of vertices are not. If there are n vertices, then the vertices can be labelled by integers i with $1 \leq i \leq n$. Then the set A of arcs can be identified by a subset of all subsets of two vertices, i.e., of subsets $\{i, j\}$, where i and j are vertices with $i \neq j$. There is an arc connecting vertices i and j if and only if the subset $\{i, j\}$ belongs to the set A. The **degree** of a vertex is the number of different arcs connected to that vertex.

One vertex is said to be a neighbor of another vertex if there is an arc connecting the two vertices. The graph is said to be **connected** if any two vertices are connected by a collection of arcs. That is, vertices i and j are connected by a collection of arcs if there is an integer k with $1 \leq k \leq n-1$ and k arcs $\{i, i_1\}$, $\{i_1, i_2\}, \ldots, \{i_{k-2}, i_{k-1}\}$, $\{i_{k-1}, j\}$.

Consider a random walk on a connected graph (a Markov chain) that moves from vertex to neighboring vertex, with each neighbor being equally likely at each move, independent of the past.

Suppose that the random walk starts at vertex 1. What is the expected total number of steps taken by the random walk until it first returns to this initial vertex?

(4) Random Walk on a Finite Weighted Graph

Consider the graph in Example 3, but let there be a weight assigned to each arc. Specifically, let there be a positive weight $w_{i,j}$ $(0 < w_{i,j} < \infty)$ assigned to the arc $\{i, j\}$ for all arcs $\{i, j\}$ in A. Again consider a random walk on a connected graph (a Markov chain) that moves from vertex to neighboring vertex, but now let the probabilities of moving to each neighboring vertex on each step be proportional to the weight on the arc connecting to that vertex.

Again suppose that the random walk starts at vertex 1. What is the expected total number of steps taken by the random walk until it first returns to this initial vertex?

2. Key facts

(1) These examples are all irreducible finite-state Markov chains.

(2) As a consequence, there is a unique stationary probability vector π, satisfying $\pi = \pi P$.

(3) Examples 1 and 2 can be regarded as a special case of Example 3, which in turn is a special case of Example 4.

(4) In Example 4 (and thus all the examples), the stationary probability vector has a very simple form:

$$\pi_i = \frac{\sum_j w_{i,j}}{\sum_i \sum_j w_{i,j}} \ .$$

(5) The simple form of the answer can be verified by just checking.

(6) The simple form of the answer can be explained by time reversibility.

(7) All these examples are time reversible, so that it suffices to solve the detailed-balance equations instead of $\pi = \pi P$, namely,

$$\pi_i P_{i,j} = \pi_j P_{j,i} \quad \textbf{for all} \quad i \quad \textbf{and} \quad j \ .$$

(8) It is easy to check that the claimed solution in (4) satisfies the detailed-balance equations in (7).

(9) It is easy to check that the the detailed-balance equations in (7) imply $\pi = \pi P$, but $\pi = \pi P$ is more general. For the implication, just sum both sides over j.

(10) In an irreducible finite-state Markov chain, the expected number of steps to first return to state i, starting in state i, is $1/\pi_i$. That property is covered by renewal theory: See Chapter 7, in particular, Proposition 7.1 in Section 7.3, for explanation. For us, the interarrival times or times between renewals are integer-valued, but that is OK. Also see Proposition 4.3 on p. 228. There we should let $r(X_n) = 1$ if $X_n = j$ and 0 otherwise.