IEOR 3106: Introduction to Operations Research: Stochastic Models

Professor Whitt, Fall 2013

Trip to the Post Office: SOLUTIONS

Five students from IEOR 3106 – Alexander Frazer (A), Yunhe Wang (Y), Bohao Zhou (B), Zhenlun Zhou (Z), and Chaitanya Kanitkar (C) – simultaneously enter an empty post office, where there are three clerks ready to serve them. Alexander (A), Yunhe (Y), Bohao (B) begin to receive service immediately, while Zhenlun (Z) and Chaitanya (C) wait in a single line, ready to be served by the first free clerk, with Zhenlun (Z) at the head of the line (to be served first when a server becomes free), and Chaitanya (C) after Zhenlun (Z). Suppose that the service times of the three clerks (for all customers) are independent exponential random variables, each with mean 2 minutes.

(a) What is the expected time (from the moment the students enter the post office) until Bohao (B) completes service?

Let A, Y, B, Z and C be random variables representing the service times of the five students, Alexander Frazer (A), Yunhe Wang (Y), Bohao Zhou (B), Zhenlun Zhou (Z), and Chaitanya Kanitkar (C), respectively. By assumption, these are independent and identically distributed (IID) exponential random variables, each with mean 2 minutes. (But these service times apply only after the students start service. Not included is the waiting time before beginning service, if any.) Since Bohao (B) starts service immediately upon arrival, the answer to this first question is simply

$$E[B] = 2 \ ,$$

with the units understood to be minutes.

(b) What is the probability that Bohao (B) is still in service after 6 minutes?

Since the distribution is exponential,

$$P(B > t) = e^{-t/2}, \quad \text{so that} \quad P(B > 6) = e^{-3} \approx 0.0498$$

(c) What is the *conditional* probability that Bohao (B) is still in service after 10 minutes, given that he has not yet been served after 4 minutes??

By the **lack-of-memory property**,

$$P(B > 10|B > 4) = P(B > 4 + 6|B > 4) = P(B > 6) = e^{-6/2} = e^{-3} \approx 0.0498,$$

same as in part (b). Note that this is easy to verify directly, just using the definition of conditional probability:

$$P(B > 10|B > 4) = \frac{P(B > 10 \quad \text{and} \quad B > 4)}{P(B > 4)} = \frac{P(B > 10)}{P(B > 4)} = \frac{e^{-10/2}}{e^{-4/2}} = e^{-(6/2)},$$

because $e^{a+b} = e^a e^b$, implying that $e^{a+b}/e^a = e^b$.

(d) What is the *conditional* probability that Bohao (B) is still in service after 10 minutes, given that Alexander (A) has not yet been served after 4 minutes??

Since Bohao (B) and Alexander (A) both start service immediately upon arrival and since the service times K and B are independent,

$$P(B > 10|A > 4) = P(B > 10) = e^{-10/2} = e^{-5} \approx 0.0067$$

(e) What is the probability that Bohao (B) is the first to complete service?

$$P(B = \min\{A, Y, B\}) = \frac{1/2}{1/2 + 1/2 + 1/2} = \frac{1}{3}$$

In general, the probability is the rate (reciprocal of the mean) for B, divided by the sum of the three rates. If the rates were λ_i with the rate for B being λ_1, then the probability would be $\lambda_1/(\lambda_1 + \lambda_2 + \lambda_3)$.

(f) What is the expected time (from the moment the students enter the post office) until the first student completes service?

The time until the first student finishes service is the minimum of the three independent exponentials. The minimum of independent exponentials is again an exponential random variable with a rate equal to the sum of the rates. The rate here for each exponential is the reciprocal of the mean, so it is $1/2$. The sum of the rates is $3/2$. Thus, the expected time is $2/3$ minute $= 40$ seconds.

(g) What is the variance of the time (from the moment the students enter the post office) until the first student completes service?

The variance of an exponential is the square of its mean. Since the mean is $2/3$, the variance is $4/9$ minute.

(h) What is the expected time (from the moment the students enter the post office) until Zhenlun (Z) completes service?

Zhenlun (Z) enters service as soon as one of the initial three finish service. Zhenlun's time in system is thus $Z + \min\{A, Y, B\}$. So the expected value is

$$E[Z + \min\{A, Y, B\}] = E[Z] + E[\min\{A, Y, B\}] = 2 + (2/3) = 8/3 \quad \text{minutes}$$

These are independent by the lack-of-memory property.

(i) What is the expected time (again since entering the post office) until *all* five students finish service?

We apply the lack-of-memory property after each service completion. Thus, the total time is the sum of five independent exponential random variables. For the first three service completions, there are three students in service simultaneously. Then there are two, and finally there is only one. When three customers are in service the rate is $3 \times (1/2) = 3/2$; When two customers are in service the rate is $2 \times (1/2) = 1$; When one customers is in service the rate is $1 \times (1/2) = 1/2$. The mean is the reciprocal of the rate. Thus the expected time is $2/3 + 2/3 + 2/3 + 1 + 2 = 5$ minutes. Notice that the expected time for the last after the first four have finished is a big part of the total mean.

(j) What is the variance of the time until *all* five students finish service?

The variance of the sum of independent random variables is the sum of the variances. The variance of each exponential random variable is the square of its mean. Thus the variance is $4/9 + 4/9 + 4/9 + 1 + 4 = 19/3$.

(k) What is the probability that Zhenlun (Z) is the *third* student to finish service?

With probability 1, Zhenlun (Z) is *not* the first student to finish. With probability 2/3, Zhenlun (Z) is *not* the second student to finish. Conditional on not being the second student to finish, with probability 1/3, Zhenlun (Z) is the third student to finish. Hence the answer is $1 \times 2/3 \times 1/3 = 2/9$.

(l) Suppose that you wanted to calculate the probability that the time required for all five students to complete service will exceed 10 minutes. What computational tool makes that calculation easy to perform? Briefly explain why.

The total time is the sum of five independent exponential random variables. If they all had the same mean, then the total time would have a gamma (or Erlang) distribution, as indicated in Section 5.2.3. However, since they do not all have the same mean, the distribution is complicated. **The desired probability is easy to compute by numerical transform inversion, however, provided (i) you can compute the Laplace transform of the cdf and (ii) you have access to an inversion program.**

Elaboration (optional.)

The time required for all five students to complete service is the sum of five independent exponential random variables, but with different means. The sum of the first three with identical means has a gamma (or Erlang) distribution, again as indicated in Section 5.2.3

on page 279. However, the sum of all five random times has a complicated distribution. The desired probability can be expressed as a multidimensional convolution integral. That multidimensional convolution integral could be computed by MATLAB. It would be made easier by reducing it to a two-dimensional integral, exploiting the gamma distribution for the first three service completions. Convolution is discussed in §2.5 on p. p. 58 of Ross; see formula (2.17). For example the density of $X+Y$, when X and Y are independent nonnegative random variables with densities f_X and f_Y is

$$f_{X+Y}(x) = \int_0^x f_X(y) f_Y(x-y)\, dy \ .$$

You just iterate those integrals to compute the density of sums of more independent random variables. The computation gets harder, though, as the number of integrals increases.

The desired computation is easy to perform, however, by numerically inverting the Laplace transform; see the papers listed on the course computational tools web page.

We give a quick brief explanation. Let X be a random variable and let f be the pdf (density) of X. The Laplace transform $\hat{f}$ is defined as

$$\hat{f}(s) \equiv E[e^{-sX}] \equiv \int_0^\infty e^{-sx} f(x)\, dx \ .$$

From elementary Laplace transform theory, the Laplace transform of the complementary cumulative distribution function (ccdf) $F^c(x) \equiv 1 - F(x)$, where the cdf is $F(x) \equiv \int_0^x f(t)\, dt$, is

$$\hat{F^c}(s) = (1 - \hat{f}(s))/s \ .$$

To be more concrete, let X be an exponentially distributed random variable with mean $1/\lambda$ (and thus rate λ). In this context, $f(x) = \lambda e^{-\lambda x}, \quad x \ge 0$. The Laplace transform here is

$$\begin{aligned} \hat{f}(s) &\equiv E[e^{-sX}] \equiv \int_0^\infty e^{-sx} f(x)\, dx \\ &= \int_0^\infty e^{-sx} \lambda e^{-\lambda x}\, dx \\ &= \lambda \int_0^\infty e^{-(s+\lambda)x}\, dx \\ &= \lambda/(\lambda + s) \ . \end{aligned}$$

Directly, the Laplace transform of the ccdf in this exponential case is

$$\begin{aligned} \hat{F^c}(s) &\equiv \int_0^\infty e^{-sx} F^c(x)\, dx \\ &= \int_0^\infty e^{-sx} e^{-\lambda x}\, dx \\ &= \int_0^\infty e^{-(s+\lambda)x}\, dx \\ &= 1/(\lambda + s) \ . \end{aligned}$$

In this exponential case we can easily verify that

$$\hat{F^c}(s) = (1 - \hat{f}(s))/s \ .$$

But why are these transforms so useful? They are useful here, as in many other cases, because the transform of $X_1 + \cdots + X_n$, say $\hat{f}$, where $X_1, \ldots, X_n$ are independent random variables such that X_i has Laplace transform $\hat{f}_i$, is

$$\hat{f}(s) = \hat{f}_1(s) \times \hat{f}_2(s) \times \cdots \times \hat{f}_n(s) \ .$$

Thus, we can easily compute the Laplace transform of the ccdf of the time required for all students to complete service. In particular, the Laplace transform is

$$\begin{aligned}
\hat{F^c}(s) &\equiv \int_0^\infty e^{-sx} F^c(x)\, dx \\
&= (1 - \hat{f}(s))/s \\
&= (1 - [\hat{f}_1(s) \times \cdots \times \hat{f}_5(s)])/s \\
&= (1 - [(\lambda_1/(\lambda_1 + s)) \times \cdots \times (\lambda_5/(\lambda_5 + s))])/s \\
&= (1 - [((3/2)/(3/2 + s))^3 \times (1/(1 + s)) \times ((1/2)/(1/2 + s))])/s \ .
\end{aligned}$$

The final expression is not so pleasing for humans to look at, but the computer is happy.

We can thus calculate the desired ccdf value $F^c(10)$ by numerically inverting that Laplace transform $\hat{F^c}(s)$. (You are not required to know how to do this.) For example, see

Joseph Abate and WW. Numerical Inversion of Laplace Transforms of Probability Distributions. *ORSA Journal on Computing*, vol. 7, 1995, pp. 36-43.

http://www.columbia.edu/ ww2040/LaplaceInversionJoC95.pdf

In particular, see algorithm EULER in Section 1 and Exhibit 1 on p. 41. For a further introductions, see §1 of

Joseph Abate, Gagan L. Choudhury and WW. An Introduction to Numerical Transform Inversion and its Application to Probability Models, in *Computational Probability*, W. Grassman (ed.), Kluwer, Boston, 1999, pp. 257-323.

http://www.columbia.edu/ ww2040/chapter.pdf

Sean Curran's Flashlight

Sean Curran has a flashlight. Sean's flashlight needs two batteries to be operational. Suppose that, in addition to his (empty) flashlight, Sean has a set of 12 functioning batteries, called battery 1, battery 2, and so forth. Initially, Sean puts batteries 1 and 2 into his flashlight, so that it starts working. Then batteries fail one by one. Whenever a battery in the flashlight fails, the flashlight stops working. Sean then tests the two batteries in the flashlight to see which one had failed, and he removes that battery. He then puts in the next available unused battery with the remaining working battery, so that the flashlight is again working. Suppose that the batteries remain like new until installed in the flashlight. Suppose that the lifetimes of the different batteries (in use in the flashlight) are independent random variables, each with an exponential distribution having a mean of 4 months. Let T be the time that the flashlight ceases to work, i.e., the time that the flashlight fails and Sean's supply of batteries is exhausted. At that moment, exactly one of the original 12 batteries will still be working. Let that last remaining working battery be battery N. Note that N is a random variable taking values in the set $\{1, 2, \ldots, 12\}$. (It will be the number of the one remaining working battery in the flashlight.)

(a) What is the expected value of T?

Whenever the flashlight is working, it has a remaining lifetime equal to the minimum of the lifetimes of the two batteries in the flashlight. By the lack-of-memory property of the exponential distribution, this lifetime is itself the minimum of two i.i.d. exponential random variables, independent of the past. Thus the remaining lifetime is an exponential random variable with rate equal to the sum of the rates for the two batteries. Since the mean lifetime of each battery is 4 months, its rate is $1/4$. The sum of two rates is thus $1/2$. The mean of that exponential random variable is the reciprocal of the rate; it is thus 2 months.

Thus the successive times to failure of the flashlight are i.i.d. exponential random variables, each with mean 2 months. The overall total life is the sum of 11 of those events. Hence,

$$E[T] = 11 \times 2 = 22 \quad \text{months.}$$

(b) What is $P(N = 12)$?

The probability that one of n independent exponential random variables is the minimum of them all is its rate divided by the sum of the rates. The last battery will be the last surviving battery if the other battery, whichever one it is, has less remaining life than battery 12 when battery 12 is installed. That is $1/2$.

(c) What is $P(N = 1)$?

For the first battery to be the last surviving battery, it would have to "win" 11 successive survival contests, each occurring independently with probability $1/2$. Thus,

$$P(N = 1) = (1/2)^{11} = \frac{1}{2048}.$$