IEOR 3106: Introduction to Operations Research: Stochastic Models
Professor Whitt, October 17, 2013

The DC Barbershop

Two enterprizing students - Nicholas Duckwiler and and Dan Caccavella - have decided to earn a little money in their spare time by opening a barbershop in their dormitory: the DC Barbershop. They have been very successful, so that they are thinking about adding another barber or possibly enlarging their waiting area. So they are applying their stochastic skills to analyze their current operation.

The class discussion is focused on Section 5 of the CTMC notes. The topic is birth-and-death processes. The focus is on models like those in Examples 5.1 and 5.2.

The current DC Barbershop has two barbers and two barber chairs, plus three extra waiting spaces. We assume that customers arriving when the system is full are blocked, leaving without receiving service or affecting future arrivals. We assume that customers arrive according to a Poisson process at rate $\lambda = 6$ per hour. We assume that the duration of each haircut is an independent exponential random variable with a mean of $\mu^{-1} = 15$ minutes. Thus the service rate of each barber is $\mu = 4$ per hour. We assume customers are served in a first-come first-served manner by the first available barber.

We also allow each waiting customer to abandon at a constant rate $\theta = 6$ per hour. Equivalently, the abandonment times for individual customers are independent exponential random variables with mean $1/\theta = 1/6$ hour or 10 minutes. (We assume customers in service do not abandon; Nicholas and Dan would not let them out of the chair.) We also allow arriving customers who would have to wait to balk immediately upon arrival. Suppose any customer who would have to wait decides to balk (leave immediately) with probability 1/3. With abandonment and balking under those assumptions, the model is still a CTMC and a BD process.

The number of customers in the DC Barbershop over time can be modeled as a continuous-time Markov chain (CTMC), specifically by a birth-and-death (BD) stochastic process; see Sections 6.3, 6.5 and 6.6 of Ross. (See exercises 6.13 and 6.14.) It is appropriate to use the transition rate form of modeling.

Let $Q(t)$ denote the number of customers in the system at time t. Then the stochastic process $\{Q(t) : t \geq 0\}$ is a BD process with six states: $0, 1, 2, 3, 4, 5$, indicating the number of customers in the system at any time. The model is specified by giving the birth rates and death rates. It is good to draw a transition rate diagram at this point, as in the last-class lecture notes. Here it has a simple form. The process moves only from a state to one of its neighbors, either up one or down one.

The **birth rates** (giving the rate of going up) are λ_i, while the **death rates** (giving the rate of going down) are μ_i. With the assumptions above, the arrival rates are $\lambda_0 = \lambda_1 = \lambda = 6$ and $\lambda_2 = \lambda_3 = \lambda_4 = 6 \times (2/3) = 4$. (The reduction is due to the balking). ($\lambda_5 = 0$, because there are no arrivals when the system is full.) The death rates are $\mu_1 = \mu = 4$, $\mu_2 = 2\mu = 8$, $\mu_3 = 2\mu + \theta = 8 + 6 = 14$, $\mu_4 = 2\mu + 2\theta = 8 + 12 = 20$, $\mu_5 = 2\mu + 3\theta = 8 + 18 = 26$. ($\mu_0 = 0$.)

The standard thing to compute is the limiting steady-state probability vector. For a CTMC, we would solve $\alpha Q = 0$ (in matrix notation), but here we have the additional BD structure. Any BD process is reversible. We solve for the steady-state probabilities, say α_i, recursively using the *local balance equations* (reversibility),

$$\alpha_i \lambda_i = \alpha_{i+1} \mu_{i+1} \quad \text{for all} \quad i \; .$$

That is, we can express α_1 directly in terms of α_0. We can then successively express α_i in terms of α_0. We then use the condition that the α_i sum over i to equal 1 in order to first find α_0 and then the other α_i.

This leads to the formula:

$$\alpha_i = \frac{r_i}{\sum_{j=0}^{j=5} r_j} \; ,$$

where $r_0 = 1$, $r_1 = \lambda_0/\mu_1$, and

$$r_j = \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \cdots \mu_j}$$

for other values of j.

Given the steady-state probability vector $\alpha \equiv (\alpha_0, \ldots, \alpha_5)$, you can then answer a variety of other questions, as we showed.

Without the balking or the abandonment, this is an $M/M/2/3$ queueing model; see Chapter 8, Sections 8.1-8.3 and 8.9. (You are not now responsible for Chapter 8.)

Typical Questions

(a) What proportion of time are both barbers busy serving customers in the long run?

Answer: $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$

(b) What is the probability that the time until the *first* customer arrives, starting empty, is greater than 10 minutes?

Answer: Note that the time until the first arrival has an exponential distribution with mean $1/\lambda = 1/6$ hour or 10 minutes. Hence, if T is the time until this arrival, then $P(T > 10) = e^{-10/10} = e^{-1}$

(c) What is the variance of the time until the *second* customer arrives, starting empty? (Suppose that time is measured in minutes.)

Answer: Recall that the time until the second arrival is the sum of two independent and identically distributed exponential random variables, each with mean 10 minutes or 1/6 hour. Since we are measuring time in minutes, the variance is $(10)^2 + (10)^2 = 200$.

(e) What proportion of all potential customers are served in the long run? (We count customers who are blocked because the system is full when they arrive and those customers who balk (refuse to join because they would have to wait) and abandon before entering as potential customers.)

Answer: The total rate of service completion is

$$\gamma \equiv \alpha_1 \mu + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)(2\mu) = 4\alpha_1 + 8(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5).$$

Since the arrival rate of potential customers is $\lambda = 6$, the proportion of all customers that are served is γ/λ.

(f) What is the expected number of customers in the shop in the long run?

Answer: $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5$