IEOR 3106: Fall 2013, Professor Whitt

Time Reversibility and Queueing Networks, Tuesday, October 29

This class is devoted to Sections 6 and 7 in the CTMC notes. See Sections 6.6 and 8.4 in Ross for additional discussion.

I. A Cafeteria OQN Example

An example of an **open network of queues (OQN)** might be a cafeteria with 7 separate stations: salad bar, sandwich bar, pizza counter, vegetarian section, hot food area, drink area and cashiers. To get a nice simple probabilistic description of the state of the system in the "long run," The model is a structured CTMC, which has nice properties.

The **model elements** are the external arrival processes, the routing probabilities, and the individual service station parameters: the number of servers and the service rate of each. For the **external arrivals**, we can assume that arrivals enter according to a Poisson process. People might then choose their first station independently according to specified probabilities. Thus somebody who only wants a drink might go directly to the drink station and then directly to the cashier, and depart before many people who arrived in the cafeteria earlier. Equivalently (by the independent thinning and superposition properties), we can have an independent external Poisson arrival process at each station. We then need to specify the arrival rate at station j, $\lambda_{e,j}$, for each station j.

Next we have the **routing**. We assume that each customer that completes service from station j goes next immediately to station k with probability $P_{j,k}$, where P is the routing matrix. The probability of leaving the network from station j, then is $P_{j,e} \equiv 1 - \sum_k P_{j,k}$.

We need to specify the station-j **service parameters**. Each service station has one or more servers. Customers who cannot be served immediately upon arrival wait in line, to be served in order of arrival at that station. We assume that station j operates separately as a $M/M/s$ queue (so that the number of customers there is a birth and death process), with s_j servers each having service rate μ_j. We assume that the service times at all stations are mutually independent exponential random variables, independent of the arrival processes The number of servers and the service rate may differ from station to station.

We might be interested in determining how many servers we need at each area. The entire system of 7 queues can also be modelled as a CTMC, but now it has special structure, so that it is possible to determine the limiting probabilities of the states for the entire system.

1. key fact 1 (another miracle provided by the exponential distribution): the limiting numbers of customers at the different stations at time t are stochastically independent. Letting ∞ denote time in the long run, we have

$$P(X_1(\infty) = k_1, \dots, X_7(\infty) = k_7) = P(X_1(\infty) = k_1) \times \cdots \times P(X_7(\infty) = k_7),$$

where we need to compute the probabilities $P(X_i(\infty) = k_i)$ properly.

2. key fact 2: the (net) arrival rate at each separate queue in the long run can be obtained by solving the traffic rate equations; see (7.1) -(7.3) in the notes.

3. key fact 3: given the net arrival rate at each queue, each separate queue is an $M/M/s$ queue, and thus can be analyzed as a birth-and-death process. That yields the formulas $P(X_i(\infty) = k_i)$. We plug these into the product form above.

4. key fact 4: The model above is an open network of queues, in which customers come from outside, mover around inside and then eventually leave. Another related model of interest is

a **closed queueing network (CQN)**. The CQN model has a fixed population of customers. Closed models are often useful to describe computer systems and manufacturing systems that tend to process a fixed number of jobs at any one time, with new jobs replacing old ones when the old ones have completed their required processing. See §8.1 in the CTMC notes.

II. Basic Theory

These are the main ideas:

1. reverse-time CTMC. (See (6.1)-(6.5).)

2. time-reversible CTMC. (See (6.6).)

3. detailed (or local) balance equations. (See (6.6).)

4. main example: birth-and-death processes. (See Theorem 6.3.)

5. the departure process from an $M/M/s$ queue. (See Theorems 6.5 and 6.6.)

6. the limiting probabilities for two $M/M/1$ queues in series. (See Theorem 6.7.)

7. an open network of queues. (See Theorem 7.1. You will *not* be responsible for the details in Section 7 beyond the statement of Theorem 7.1 or the proofs anywhere.)

III. Covered Material in the CTMC Notes

Here is a summary of the material we covered in the CTMC notes. We covered §2, §3.1 and §3.2, §4, the statements of Theorems 5.1 and 5.2 (b) and (c), §6 through Theorem 6.7 and §7 through Theorem 7.1.