IEOR 3106, Fall 2013, Professor Whitt

Introduction to Renewal Theory: Tuesday, November 12

I. Renewal Processes, §7.1

A *renewal counting process* $\{N(t) : t \geq 0\}$ is a generalization of a Poisson counting process, in which the intervals between successive points X_n are i.i.d., but not necessarily exponentially distributed. Even when the exponential assumption is dropped, there is a lot we can do.

Suppose that we have a sequence of i.i.d. random variables $\{X_n : n \geq 1\}$, where $P(X_1 > 0) = 1$. The random variable variable X_n is regarded as the *interrenewal time* between the $(n-1)^{\mathrm{st}}$ and n^{th} renewals in a renewal process; i.e., there is an associated *renewal counting process* $\{N(t) : t \geq 0\}$ with

$$N(t) \equiv \sup\{n \geq 0 : S_n \leq t\}, \quad t \geq 0,$$

where S_n is the n^{th} partial sum from $\{X_n\}$, i.e.,

$$S_n \equiv X_1 + \cdots X_n, \quad n \geq 1 \ ,$$

with $S_0 \equiv 0$; see Chapter 7 in Ross.

II. Basic Renewal-Reward Theory, §7.4

Now we assume that there ia a reward associated with each renewal. The successive rewards are i.i.d. In particular, suppose that we have a sequence of i.i.d. random vectors $\{(X_n, R_n) : n \geq 1\}$, where $P(X_1 > 0) = 1$. The random variable variable X_n is regarded as the *interrenewal time* between the $(n-1)^{\mathrm{st}}$ and n^{th} renewals in a *renewal counting process* $\{N(t) : t \geq 0\}$. We think of the random variables R_n as rewards. We "earn" reward R_n at time S_n. We define an associated continuous-time *renewal-reward stochastic process* by setting

$$R(t) \equiv \sum_{n=1}^{N(t)} R_n, \quad t \geq 0 \ .$$

The stochastic process $\{R(t) : t \geq 0\}$ is the renewal-reward process. The random variable $R(t)$ is the cumulative reward earned up to time t; see Section 7.4 of Ross. When the renewal process is a Poisson process, the renewal reward process becomes a compound Poisson process.

We can do a surprising amount with the law of large numbers (LLN) for renewal-reward processes. This is perhaps the easiest part of renewal theory and yet it is perhaps the most useful part. So we start by studying this part.

Theorem 0.1 (*SLLN for renewal-reward processes*) *If $E[X_1] < \infty$ and $E[|R_1|] < \infty$, then*

$$\frac{R(t)}{t} \rightarrow \frac{E[R_1]}{E[X_1]} \ .$$

It has an easy proof, which we will go through. See Proposition 7.3 of Ross. It draws on the SLLN for the renewal counting process $\{N(t) : t \geq 0\}$; see Proposition 7.1 in Ross.

Theorem 0.2 (*SLLN for renewal counting process*) *If $E[X_1] < \infty$, then*

$$\frac{N(t)}{t} \rightarrow \frac{1}{E[X_1]} \ .$$

The SLLN for renewal processes draws on the classical SLLN for sums of i.i.d. random variables, Theorem 2.1 on page 79. These SLLN's require that the random variables involved have finite means. (The word "strong" means that the limit holds "with probability 1," as opposed to some weaker notion. The weak law of large numbers (WLLN) involves convergence in probability or convergence in distribution, a weaker notion. We do not focus on the issue of the mode of convergence.) The two propositions in Section 7 are very easily proved, given the classic result stated without proof in Chapter 2. Those proofs are given in the book, and were done in class.

What that theory concludes is that **the long-run average reward over time is the expected reward per cycle, divided by the expected length of the cycle**, where a cycle is the time between successive renewals. As in Proposition 7.3, the total reward earned up to time t is defined as

$$R(t) \equiv \sum_{i=1}^{N(t)} R_i, \quad t \geq 0,$$

where R_i is the reward in cycle i and $\{N(t) : t \geq 0\}$ is the renewal counting process with X_i as the time between renewals $i-1$ and i. The SLLN for renewal-reward processes states that

$$\lim_{t \to \infty} \frac{R(t)}{t} = \frac{E[R_1]}{E[X_1]}.$$

III. Examples

1. Small-Town Traffic Cop: Speeding Ticket Revenue

A small-town traffic cop spends his entire day on the lookout for speeders. The policeman cruises on average approximately 10 minutes before stopping a car for some offense. Of the cars he stops, 90% of the drivers are given speeding tickets with an \$80 fine. It takes the policeman an average of 5 minutes to write such a ticket. The other 10% of the stops are for more serious offenses, leading to an average fine of \$300. These more serious charges take an average of 30 minutes to process. In the long run what is the rate of money brought in by fines?

ANSWER

The average time between successive stops is

$$10 + (0.9 \times 5) + (0.1 \times 30) = 10 + 4.5 + 3.0 = 17.5 \quad \text{minutes} .$$

The average fine revenue per stop is

$$0.9 \times 80 + 0.1 \times 300 = 72 + 30 = 102$$

Hence, by the renewal reward theorem, the long-run average rate fine revenue is accrued is

$$\frac{E[\text{fine per stop}]}{E[\text{time per stop}]} = \frac{\$102}{17.5 \quad \text{minutes}} = \$5.28 \quad \text{per minute} .$$

Not a bad job!

2. Long-haul Truck Driver: Driving Back and Forth

A truck driver continually drives from Atlanta (A) to Boston (B) and then immediately back from B to A. Each time he drives from A to B, he drives at a fixed speed that (in miles per hour) is randomly chosen, uniformly in the interval $[40, 60]$. Each time he drives from B to A, he drives at a fixed speed that (in miles per hour) is randomly chosen, being either 40 or 60, each with probability 1/2. (Going from B to A the random speed has a Bernoulli distribution.) The successive random experiments at the beginning of each trip are mutually independent.

(a) In the long run, what proportion of his driving time is spent driving from A to B?

(b) In the long run, what proportion of his driving time is spent driving at a speed of 40 miles per hour?

ANSWERS

The key idea has nothing to do with this course: You should remember **dirt**. If you can't remember this, you are **" dumb as dirt"**, to quote Robin Williams in his recording of Pecos Bill (another weird reference).

Dirt means $\mathbf{D} = \mathbf{RT}$, *distance equals rate times time*. Here we are given random speeds or rates, but we ask about times. So whatever the distance D, the time T is $T = D/R$. We must thus average the reciprocal of the rate. That explains the solution. Note that $E[1/R]$ is not $1/E[R]$. That explains why the two expected times are not equal.

(a) The proportion of his driving time spent driving from A to B is

$$\frac{E[T_{A,B}]}{E[T_{A,B}] + E[T_{B,A}]} \, ,$$

where $E[T_{A,B}]$ is the expected time to drive from A to B, while $E[T_{B,A}]$ is the expected time to drive from B to A.

To find $E[T_{A,B}]$ and $E[T_{B,A}]$, we use the elementary formula $d = rt$ (distance = rate $\times$ time). Let S be the driver's random speed driving from A to B. Then

$$\begin{aligned} E[T_{A,B}] &= \frac{1}{20}\int_{40}^{60} E[T_{A,B}|S = s]\, ds \\ &= \frac{1}{20}\int_{40}^{60} \frac{d}{s}\, ds \\ &= \frac{d}{20}(ln(60) - ln(40)) \\ &= \frac{d}{20}(ln(3/2) \, . \end{aligned}$$

Similarly,

$$\begin{aligned} E[T_{B,A}] &= \frac{1}{2}E[T_{B,A}|S = 40] + \frac{1}{2}E[T_{B,A}|S = 60] \\ &= \frac{1}{2}(\frac{d}{40} + \frac{d}{60}) \\ &= \frac{d}{48} \end{aligned}$$

(b) Assume that a reward is earned at rate 1 per unit time whenever he is driving at a rate of 40 miles per hour, we can again apply the renewal reward approach. If p is the long-run proportion of time he is driving 40 miles per hour,

$$p = \frac{(1/2)d/40}{E[T_{A,B}] + E[T_{B,A}]} = \frac{1/80}{\frac{1}{20}ln(3/2) + 1/48} \ .$$