IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2013, Professor Whitt,

More Renewal Theory, Chapter 7, Thursday, November 14

1. The Key Formula

The key formula is

$$\begin{aligned} r &\equiv \text{the long run average reward} \equiv \lim_{t\to\infty} \frac{R(t)}{t} \\ &= \frac{E[R]}{E[X]} = \frac{\text{the average reward per cycle}}{\text{the average length of a cycle}}, \end{aligned}$$

where $R(t)$ is the total reward earned up to time t, X is a generic cycle (time between renewals) and R is a generic reward in one cycle. This can be justified quite directly by the law of large numbers; see the last lecture notes.

We assume that $\{(X_n, R_n) : n \geq 1\}$ is a sequence of i.i.d random vectors. We allow X and R to be dependent, as was the case with the traffic cop example. The time to write a ticket was correlated with the dollar value of the ticket.)

2. the Inspection Paradox, the Excess and the Age

We discussed the inspection paradox and the excess and age of a renewal process; see Section 7, Examples 7.16 and 7.17 and Examples 7.23 and 7.24. The age process and the (forward) excess process associated with a renewal process are shown in Figure 1. We discussed the inspection paradox, which is presented in Section 7.7 of the book. Homework exercise 51 is related to this. (That is the study of tourism in Morocco.) We talked about how long a person would have to wait for the next bus, if he were to go out to the bus stop at a random time, unaware of the bus schedule. We emphasized that the distribution of the time until the next bus, from the perspective of an arrival at an arbitrary time, is in general different than the distribution of the time between successive bus arrivals. This is true in great generality. For this chapter, though, we assume that the successive bus arrivals form a renewal process, which means that the successive intervals between successive arrivals form a sequence of IID random variables, each distributed as the random variable X with cdf F.

Let F be the cdf of the time between successive arrivals. Suppose that the cdf F has mean m. Then the cdf of the time until the next bus, from the perspective of an arrival at an arbitrary time, is

$$F_e(x) = \frac{1}{m} \int_0^x (1 - F(u))\, du \ ;$$

see Examples 7.23 and 7.24 in the book. The cdf F_e is the *equilibrium-excess cdf* or the stationary-excess cdf or the equilibrium-residual-lifetime cdf. That distribution is the same as the distribution of the age, discussed in Example 7.23.

To analyze the age distribution, it is convenient to use the tail-integral expression for the mean:

$$E[X] = \int_0^\infty P(X > y)\, dy$$

see bottom of p. 610 (p. 604 in the ninth edition, p. 580 of the eighth edition), and see the final section here.

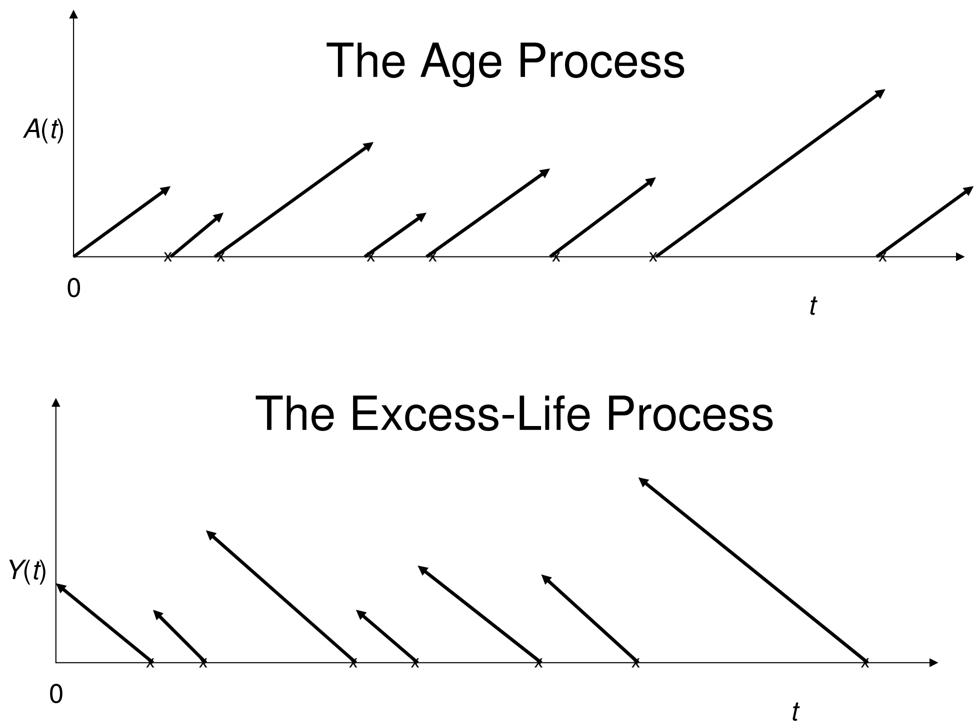

Figure 1: The age and residual-lifetime processes associated with a renewal process.

We might just be interested in the mean of the cdf F_e. In other words, we might be interested in the long-run average. The average age and average excess (or residual life or remaining life) are discussed in Example 7.15. and Example 7.16. Note that the average age and average excess depends on the second moment of the time between successive points, as well as the mean, as given in formula (7.14). I suggested rewriting (7.14) as

$$\text{average value of excess} = \text{average age} = \frac{E[X^2]}{2E[X]} = \frac{(c^2+1)}{2} \times E[X] \ ,$$

where

$$c^2 \equiv \frac{Var(X)}{E[X]^2}$$

is the *squared coefficient of variation* (SCV). The SCV measures variability independent of the mean or, equivalently, independent of scale. Note that

$$c^2_{bX} = c^2_X$$

for any positive constant b and any nonnegative random variable X. For an exponential random variable X, $c^2_X = 1$; for a deterministic random variable (P(X=d) = 1), $c^2_X = 0$.

3. A Computer with Three parts

Paul's computer has three critical parts, each of which is needed for the computer to work. The computer runs continuously as long as the three required parts are working. The three parts have mutually independent exponential lifetimes before they fail. The expected lifetime of parts 1, 2 and 3 are 10 weeks, 20 weeks and 30 weeks, respectively. When a part fails, the computer is shut down and an order is made for a new part of that type. When the computer is shut down (to order a replacement part), the remaining two working parts are not subject to failure. The time to replace part 1 is exponentially distributed with mean 1 week; the time to replace part 2 is uniformly distributed between 1 week and 3 weeks; and the time to replace part 3 has a gamma distribution with mean 3 weeks and standard deviation 10 weeks.

(a) Assuming that all parts are initially working, what is the expected time until the first part fails?

Let T be the time until the first failure. Then T is exponential with rate equal to the sum of the rates; see Chapter 5; i.e.,

$$ET = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{(1/10) + (1/20) + (1/30)} = \frac{1}{(11/60)} = \frac{60}{11} = 5.45 \text{ weeks} .$$

(b) What is the probability that part 1 is the first part to fail?

Let N be the index of the first part to fail. Since the failure times are mutually independent exponential random variables (see Chapter 5),

$$P(N = 1) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{(1/10)}{(1/10) + (1/20) + (1/30)} = \frac{6}{11} = 0.545 .$$

(c) What is the long-run proportion of time that the computer is working?

Now for the first time we need to consider the random times it takes to get the replacement parts. Actually these distributions beyond their means do not affect the answers to the questions asked here. Only the means matter here. Use elementary renewal theory. The successive times that the computer is working and shut down form an alternating renewal process. Or, equivalently, apply renewal reward processes, as in Section 7.4: Look at the expected reward per cycle divided by the expected length of a cycle. Let a reward be earned at rate 1 whnenver the computer is working. Let T be a time until a failure (during which the computer is working) and let D be a down time. A cycle is $T + D$. Then the long-run proportion of time that the computer is working is

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t R(s)\, ds = \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{N(t)} T_i = \frac{ET}{ET + ED},$$

where $N(t)$ is the renewal counting process counting the number of complete cycles up to time t. (This ignores the last cycle in process at time t, but the contribution of it is asymptotically negligible.)

By part (a) above, $ET = 60/11$. It suffices to find ED. To find ED, we consider the three possibilities for the part that fails:

$$\begin{aligned} ED &= P(N=1)E[D|N=1] + P(N=2)E[D|N=2] + P(N=3)E[D|N=3] \\ &= (6/11)E[D|N=1] + (3/11)E[D|N=2] + (2/11)E[D|N=3] \\ &= (6/11)1 + (3/11)2 + (2/11)3 \\ &= 18/11 . \end{aligned}$$

Hence,

$$ET/(ET+ED) = \frac{(60/11)}{(60/11)+(18/11)} = \frac{60}{78} = \frac{30}{39} \approx 0.769$$

(d) Suppose that new parts of type 1 each cost \$50; new parts of type 2 each cost \$100; and new parts of type 3 each cost \$400. What is the long-run average cost of replacement parts per week?

This is yet another application of the renewal reward theorem. The cycle is the same as before, but now we have a new reward R. Now

$$\begin{aligned}
\frac{ER}{EC} &= \frac{ER}{ET+ED} \\
&= \frac{P(N=1)50+P(N=2)100+P(N=3)400}{78/11} \\
&= \frac{(6/11)50+(3/11)100+(2/11)400}{78/11} \\
&= \frac{(300/11)+(300/11)+(800/11)}{78/11} \\
&= \frac{1400/11}{78/11} = \frac{1400}{78} \approx 17.95 \text{ dollars per week}
\end{aligned}$$

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4. The Tail Integral Formula for the Mean

We digress to give a derivation of the tail integral formula for the mean; see footnote on p. 610.

Lemma 0.1 *For a nonnegative random variable X with cdf F having pdf f, the mean is*

$$E[X] = \int_0^\infty x f(x)\,dx = \int_0^\infty P(X>x)\,dx = \int_0^\infty (1-F(x)\,dx.$$

Proof. Since

$$P(X>x) = 1-F(x) = \int_x^\infty f(y)\,dy,$$

$$\begin{aligned}
\int_0^\infty P(X>x)\,dx &= \int_0^\infty \left(\int_x^\infty f(y)\,dy\right) dx \\
&= \int_0^\infty \int_0^y f(y)\,dx\,dy = \int_0^\infty f(y)\left(\int_0^y dx\right) dy \\
&= \int_0^\infty x f(x)\,dx,
\end{aligned}$$

which is justified by (carefully) changing the order of the integration in line 2.