IEOR 3106: Fall 2013, Professor Whitt

Topics for Discussion: Tuesday, November 19

Alternating Renewal Processes and The Renewal Equation

1 Alternating Renewal Processes

An alternating renewal process alternates between two states, which we call "up" and "down," thinking of the next example in which a computer is either working (up) or not working (down). The system alternates between successive up intervals and down intervals. The system is first up for a time U_1 and then down for a period D_1. Next it is up for a period U_2 and then down for a period D_2, and so on. Often, the sequences of up times $\{U_n : n \geq 1\}$ and down times $\{D_n : n \geq 1\}$ are independent sequences of i.i.d. nonnegative random variables. However, as in the traffic cop example and the computer example below, all we really require is that the sequences of pairs $\{(U_n, D_n) : n \geq 1\}$ be i.i.d. random vectors (pairs of nonnegative random variables). We allow U_n and D_n to be dependent for each n, but different pairs are i.i.d.

Let U be a generic up time; let D be a generic down time. The distribution of U need not be the same as the distribution of D. Let $Z(t)$ be the state at time t; i.e., Let $Z(t) = 1$ if the alternating renewal process is up at time t, starting at time 0 at the beginning of an up period; otherwise let $Z(t) = 0$. We can apply the renewal reward theorem to deduce that the long-run proportion of time that the alternating renewal process is up is

$$\text{long-run proportion up} \quad = \lim_{t \to \infty} \frac{1}{t} \int_0^t Z(s)\, ds = \frac{E[U]}{E[U] + E[D]}. \tag{1}$$

2 An Example: ZeLun Gu's Computer

Zelun's computer has three critical parts, each of which is needed for the computer to work. The computer runs continuously as long as the three required parts are working. The three parts have mutually independent exponential lifetimes before they fail. The expected lifetime of parts 1, 2 and 3 are 10 weeks, 20 weeks and 30 weeks, respectively. When a part fails, the computer is shut down and an order is made for a new part of that type. When the computer is shut down (to order a replacement part), the remaining two working parts are not subject to failure. The time to replace part 1 is exponentially distributed with mean 1 week; the time to replace part 2 is uniformly distributed between 1 week and 3 weeks; and the time to replace part 3 has a gamma distribution with mean 3 weeks and standard deviation 10 weeks.

(a) Assuming that all parts are initially working, what is the expected time until the first part fails?

Let T be the time until the first failure. Then T is exponential with rate equal to the sum of the rates; see Chapter 5; i.e.,

$$ET = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{(1/10) + (1/20) + (1/30)} = \frac{1}{(11/60)} = \frac{60}{11} = 5.45 \text{ weeks} .$$

(b) What is the probability that part 1 is the first part to fail?

Let N be the index of the first part to fail. Since the failure times are mutually independent exponential random variables (see Chapter 5),

$$P(N=1) = \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} = \frac{(1/10)}{(1/10)+(1/20)+(1/30)} = \frac{6}{11} = 0.545 \ .$$

(c) What is the long-run proportion of time that the computer is working?

Now for the first time we need to consider the random times it takes to get the replacement parts. Actually these distributions beyond their means do not affect the answers to the questions asked here. Only the means matter here. Use elementary renewal theory. The successive times that the computer is working and shut down form an alternating renewal process. Or, equivalently, apply renewal reward processes, as in Section 7.4: Look at the expected reward per cycle divided by the expected length of a cycle. Let a reward be earned at rate 1 whnenver the computer is working. Let T be a time until a failure (during which the computer is working) and let D be a down time. A cycle is $T+D$. Then the long-run proportion of time that the computer is working is

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t R(s)\,ds = \lim_{t\to\infty} \frac{1}{t}\sum_{i=1}^{N(t)} T_i = \frac{ET}{ET+ED},$$

where $N(t)$ is the renewal counting process counting the number of complete cycles up to time t. (This ignores the last cycle in process at time t, but the contribution of it is asymptotically negligible.)

By part (a) above, $ET = 60/11$. It suffices to find ED. To find ED, we consider the three possibilities for the part that fails:

$$\begin{aligned} ED &= P(N=1)E[D|N=1] + P(N=2)E[D|N=2] + P(N=3)E[D|N=3] \\ &= (6/11)E[D|N=1] + (3/11)E[D|N=2] + (2/11)E[D|N=3] \\ &= (6/11)1 + (3/11)2 + (2/11)3 \\ &= 18/11 \ . \end{aligned}$$

Hence,

$$ET/(ET+ED) = \frac{(60/11)}{(60/11)+(18/11)} = \frac{60}{78} = \frac{30}{39} \approx 0.769$$

(d) Suppose that new parts of type 1 each cost \$50; new parts of type 2 each cost \$100; and new parts of type 3 each cost \$400. What is the long-run average cost of replacement parts per week?

This is yet another application of the renewal reward theorem. The cycle is the same as before, but now we have a new reward R. Now

$$\frac{ER}{EC} = \frac{ER}{ET+ED}$$

$$
\begin{aligned}
&= \frac{P(N=1)50 + P(N=2)100 + P(N=3)400}{78/11} \\
&= \frac{(6/11)50 + (3/11)100 + (2/11)400}{78/11} \\
&= \frac{(300/11) + (300/11) + (800/11)}{78/11} \\
&= \frac{1400/11}{78/11} = \frac{1400}{78} \approx 17.95 \text{ dollars per week}
\end{aligned}
$$

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3 Harder Results for Alternating Renewal Processes

We now want to establish a harder (more refined) result (needing extra conditions). Now we want to show that

$$\lim_{t\to\infty} P(Z(t) = 1) = \frac{E[U]}{E[U] + E[D]}. \tag{2}$$

The answer in (2) is the same as in (1), but the form of the limit is stronger. For example, suppose that all up times are of length 1 and all down times are of length 2. Then

$$P(Z(3k+0.5) = 1) = 1 \quad \text{while} \quad P(Z(3k+1.5) = 1) = 0 \quad \text{for all} \quad k \geq 1 \quad \text{integer.}$$

One subsequence has one limit, while another subsequence has a different limit. We will need to impose extra conditions that rule out this case. We will do so by assuming that the distribution of a cycle length is **nonlattice**. That means that the cdf of X, say F, does not concentrate all its mass on the integers or a subset $\{cn : n \geq 1\}$; the integer case is the most common lattice distribution; then $c = 1$. Any nonnegative random variable with a probability density function has a nonlattice distribution. Any integer-valued discrete distribution has a lattice distribution.

Let $P(t) = P(Z(t) = 1)$ be the probability that an up-down alternating renewal process, starting at time 0 at the beginning of an up period, is in the up state at time t. We obtain an explicit expression for $P(t)$ by setting up a **renewal equation**. We then construct the unique **solution to the renewal equation** and apply the **key renewal theorem** to obtain the desired limit in (2).

3.1 The Renewal Equation for an Alternating Renewal Process

Let $H(t) \equiv P(U \leq t)$ and $F(t) \equiv P(X \leq t)$, where $X = U + D$ is a generic inter-renewal time. (We do not need the cdf of D or the two-dimensional cdf of (U, D).) Let $H^c(t) \equiv 1 - H(t)$ be the complementary cdf (ccdf). We express $P(t)$ by conditioning on the value of X_1 and unconditioning:

$$
\begin{aligned}
P(t) &\equiv P(Z(t) = 1) \\
&= P(Z(t) = 1, X_1 > t) + P(Z(t) = 1, X_1 \leq t) \\
&= P(U_1 > t) + \int_0^t P(Z(t) = 1 | X_1 = s) f_{X_1}(s)\, ds, \\
&= P(U_1 > t) + \int_0^t P(Z(t-s) = 1) f_{X_1}(s)\, ds, \\
&= H^c(t) + \int_0^t P(t-s) f(s)\, ds. \qquad (3)
\end{aligned}
$$

Line one above is the definition. In line two above we use the law of total probability: $P(A) = P(A \cap B) + P(A \cap B^c)$. For line three, we observe that the event $\{Z(t) = 1, X_1 > t\}$ is equivalent to the event $\{U_1 > t\}$ for the first term. Hence, they necessarily have the same probability. For the second term on line three, we use the formula $P(AB) \equiv P(A \cap B) = P(A|B)P(B)$. In line four of (3), we used the renewal property,

$$P(Z(t) = 1|X_1 = s) = P(Z(t-s) = 1) = P(t-s) \quad \text{for all} \quad s, \quad 0 \le s \le t.$$

The final display in (3) is the a renewal (integral) equation, i.e.

$$P(t) = H^c(t) + \int_0^t P(t-s) f(s) \, ds \ , \tag{4}$$

This renewal equation in (4) is valid for all $t \ge 0$. Equation (4) is an integral functional equation, i.e., an equation for the function $P(t)$ regarded as a function of $t \ge 0$. Note that the function $\{P(t) : t \ge 0\}$ appears on both sides of the equation. To obtain an explicit expression for the function $P(t)$, we must solve the equation. The first term in (4) describes the probability associated with the first renewal interval. The second term describes what happens after time s and before time t if s is the time that the first cycle ends.

3.2 Solving the Renewal Equation and Establishing the Limit

We now show how to solve the renewal equation. We recall the definition of a renewal process and we introduce the notion of the renewal function.

3.2.1 A Renewal Process

Let $\{X_n : n \ge 1\}$ be a sequence of IID nonnegative real-valued random variables with cdf F having finite mean EX. Avoid trivialities by assuming that $P(X > 0) > 0$. Let

$$S_n \equiv X_1 + \cdots + X_n, \quad n \ge 1 \ , \tag{5}$$

with $S_0 \equiv 0$ and

$$N(t) \equiv \max \{n \ge 0 : S_n \le t\}, \quad t \ge 0 \ . \tag{6}$$

Then $N \equiv \{N(t) : t \ge 0\}$ is a *renewal (counting) process*, for which X_n is the length of the interval between the $(n-1)^{\text{st}}$ point and the n^{th} point, for $n \ge 1$. (We do not put a point at 0 unless $X_1 = 0$.)

In addition, for the asymptotic results we will want to assume that the distribution of X_1 is non-lattice. A distribution is a lattice distribution if it concentrates all mass (has support) on the sequence $\{cn : n \ge 0\}$ for some constant $c > 0$. The most common lattice distributions have mass concentrated on the integers (the case $c = 1$). For lattice distributions, there are corresponding limits, along the lattice of support.

3.2.2 The Renewal Function

Then the *renewal function* is the mean of the renewal process $N(t)$ as a function of t, i.e.,

$$m(t) \equiv E[N(t)], \quad t \ge 0 \ . \tag{7}$$

We have the following expression for the renewal function:

$$m(t) \equiv E[N(t)] = \sum_{n=1}^{\infty} P(N(t) \ge n) = \sum_{n=1}^{\infty} P(S_n \le t) = \sum_{n=1}^{\infty} F_n(t) \ , \tag{8}$$

where $S_n \equiv X_1 + \cdots + X_n$ and $F_n(t) \equiv P(S_n \le t)$.

3.2.3 The Solution to the Renewal Equation in (4)

It is significant the we can solve the functional equation (4) for the function $\{P(t) : t \geq 0\}$. We now describe the results, but not yet the derivation. Equation (4) has solution (as can be seen by taking transforms; see the §5.2 below)

$$P(t) = H^c(t) + \int_0^t H^c(t-s)\, dm(s) \ , \tag{9}$$

where $m(t) = E[N(t)]$ is the **renewal function** associated with the renewal counting process $N(t)$, counting the number of completed cycles up to time t.

If $m(t)$ has a derivative with respect to t, denoted by $m'(t) = \nu(t)$, then $dm(s) = \nu(s)\, ds$ in the integral in (9). Consider that case if you are not familiar with the Riemann-Stieltjes integral with $dm(s)$. Then we write the expression in (9) as a usual integral

$$P(t) = H^c(t) + \int_0^t H^c(t-s) m'(s)\, ds = H^c(t) + \int_0^t H^c(t-s)\nu(s)\, ds \ . \tag{10}$$

Notice that in (9) the function $\{P(t) : t \geq 0\}$ now only appears on the left. Equation (9) gives an explicit expression for $P(t)$. We go from (4) to (9) by replacing the two terms inside the integral: Inside the integral, we replace P by H^c and we replace the density $f(s)ds$ by $dm(s) = \nu(s)ds$.

3.2.4 Determining the Desired Limit

The standard application of the renewal equation (4) and its solution (9) or (10) is to drive the limit of the function on the left as $t \to \infty$. That is, we apply (10) to derive the limit $\lim_{t\to\infty} P(t)$. We need regularity conditions. First, we need the renewal intervals X_n to have a nonlattice distribution. That means that the cdf of X, which is F, does not concentrate all its mass on the integers or a subset $\{cn : n \geq 1\}$; the integer cases is the most common lattice distribution; then $c = 1$. And nonnegative random variable with a probability density function has a nonlattice distribution. Any integer-valued discrete distribution has a lattice distribution.

We also need other regularity conditions. The main theorem is the **key renewal theorem**. It concludes that, if regularity conditions are satisfied, including the cdf F being nonlattice, then the solution of the renewal equation converges to a limit.

Theorem 3.1 (*key renewal theorem for alternating renewal process*) *If the interval between renewals X has a finite mean and a nonlattice distribution, then the function $P(t)$ on the left of* (4) *and* (9) *converges to a proper limit as $t \to \infty$, in particular,*

$$\lim_{t\to\infty} P(t) = \frac{\int_0^\infty H^c(s)\, ds}{E[X]}.$$

Let us consider the specific case of an alternating renewal process. Here is what we expect. By the elementary renewal theorem, we have $m(t)/t \to \lambda \equiv 1/E[X]$ as $t \to \infty$, where $E[X] = E[U] + E[D]$ Hence, we expect that $\nu(t) = m'(t) \to \lambda$ as $t \to \infty$. Clearly, $H^c(t) \to 0$ as $t \to \infty$. By the tail integral formula for the mean, the integral of H^c is the mean $E[U]$, where U has cdf H. Thus,

$$\int_0^t H^c(t-s)\, ds = \int_0^t H^c(s)\, ds \to \int_0^\infty H^c(s)\, ds = E[U] \quad \text{as} \quad t \to \infty \ . \tag{11}$$

Thus we see that the key renewal theorem states what we expect:

$$\int_0^t H^c(t-s)\nu(s)\,ds = \int_0^t H^c(s)\nu(t-s)\,ds \to \lambda \int_0^\infty H^c(s)\,ds\ . \tag{12}$$

As a consequence, in the specific case of an alternating renewal process, we get

$$\begin{aligned} P(t) &= H^c(t) + \int_0^t H^c(t-s)\,dm(s) = H^c(t) + \int_0^t H^c(s)\,dm(t-s) \\ &\to 0 + \lambda \int_0^\infty H^c(s)\,ds = \frac{E[U]}{E[U]+E[D]}\ . \end{aligned} \tag{13}$$

In the last line, $1/\lambda$ is the mean cycle length $E[U]+E[D]$, so that $\lambda = 1/(E[U]+E[D])$, while $E[U] = \int_0^\infty H^c(s)\,ds$, by the tail integral formula for the mean.

Many applications can be treated by this argument. The challenging careful mathematical treatment finds precise conditions for all of the above to be fully rigorous. The book, *Applied Probability and Queues* by S. Asmussen, second edition, Springer, 2003, does a nice job on that, but it is at a level beyond this course.

4 Easy and Hard Renewal Theory for the Age Process

We now give more general background, including a treatment of the age, discussed in a previous class.

Very broadly, the word "renewal" connotes "starting over." Renewal theory involves only a few key ideas: First, renewal theory is about **renewal processes**. A key quantity associated with a renewal process is the **renewal function**. The renewal function is important because it is a key component of the solution of the **renewal equation**. We use the renewal equation to solve harder problems in renewal theory. The "easy" renewal theory concerns application of the renewal reward theorem, yielding long-run averages or long-run proportions.

4.1 Easy Renewal Theory

For example, when we consider the age process, the easy renewal theory tells us that the long-run proportion of time that the age $A(t)$ is less than or equal to a constant c is

$$F_e(c) \equiv \frac{1}{E[X]} \int_0^c F^c(t)\,dt, \tag{14}$$

where X is an interval between renewals, having cdf $F(x) \equiv P(X \le x)$ and F^c is the complementary cdf, i.e., $F^c(x) \equiv 1 - F(x)$. The tail integral formula for the mean implies that F_e is a bonafide cdf. The cdf F_e is called the stationary (or equilibrium) age distribution. From the easy renewal theory (notes last time), we conclude that

$$\frac{\int_0^t 1_{\{A(s)\le c\}}(s)\,ds}{t} \to \frac{E[\int_0^X 1_{\{A(t)\le c\}}\,dt]}{E[X]} = F_e(c) \quad \text{as} \quad t \to \infty. \tag{15}$$

4.2 Harder Renewal Theory

In contrast, the "harder" renewal theory shows, under general conditions, that the following stronger limit than (15) is valid

$$P(A(t) \le c) \to F_e(c) \quad \text{as} \quad t \to \infty. \tag{16}$$

In general, the limit (16) implies the limit (15), whereas the limit (15) does not imply the limit (16). Of course, the actual value of the limit is the same in both cases. Thus, to a large extent, the easy renewal theory already provides most of the information.

The harder renewal theory involves the asymptotic form of the renewal function and the solution of the renewal equation. Several theorems describe this asymptotic behavior: (i) the **elementary renewal theorem**, (ii) **Blackwell's renewal theorem**, and (iii) **the key renewal theorem**. These notes explain.

5 Using Laplace Transforms (OPTIONAL FROM HERE ON)

A relatively easy way to solve the renewal equations is to apply Laplace transforms, but some care is needed, because the solution of the renewal equation is expressed in terms of $dm(t)$ as opposed to $m(t)$. We explain below.

For a real-valued function g of a nonnegative real variable, its Laplace transform is defined as

$$\hat{g}(s) \equiv \int_0^\infty e^{-st} g(t)\, dt,$$

where in general s is a complex variable with positive real part, but it suffices to think of s as a positive real number.

5.1 Laplace Transform of a Random Variable

For any nonnegative random variable Z with probability density function (pdf) g, its Laplace transform is defined as

$$\hat{g}(s) \equiv E[e^{-sZ}] = \int_0^\infty e^{-st} g(t)\, dt.$$

That is, the Laplace transform of the random variable is understood to be the ordinary Laplace transform of its density (pdf).

5.2 The Laplace Transform of the Renewal Function

We now develop an expression for the Laplace transform of $m(t)$, i.e., for

$$\hat{m}(s) \equiv \int_0^\infty e^{-st} m(t)\, dt.$$

From above, the Laplace transform of the cdf $F_n(t)$ (of S_n) appearing in (8) is

$$\hat{F}_n(s) \equiv \int_0^\infty e^{-st} F_n(t)\, dt. \tag{17}$$

We now develop further expressions. First, we write the Laplace transform of the density directly as

$$\hat{f}_n(s) \equiv \int_0^\infty e^{-st} f_n(t)\, dt. \tag{18}$$

Since the cdf $F_n(t)$ is directly the integral of the pdf $f_n(t)$, basic Laplace transform theory tells us that formulas (17) and (18) imply that

$$\hat{F}_n(s) = \frac{\hat{f}_n(s)}{s}. \tag{19}$$

(The relation (19) follows by applying integration by parts.)

Next, because $S_n \equiv X_1 + \cdots + X_n$, where the random variables X_i are i.i.d., the pdf f_n is the n-fold convolution of the pdf f_1 with itself. (See below.) Thus, we have the following important simplifying power relation

$$\hat{f}_n(s) \equiv \hat{f}(s)^n. \tag{20}$$

We thus can combine formulas (17)-(20) to obtain

$$\hat{F}_n(s) \equiv \int_0^\infty e^{-st} F_n(t)\, dt. = \frac{\hat{f}_n(s)}{s} = \frac{\hat{f}(s)^n}{s} \ , \tag{21}$$

where $\hat{f}_n(s)$ is the Laplace transform of f_n, the pdf associated with the cdf F_n, and $\hat{f}(s) = \hat{f}_1(s)$ is the Laplace transform of the pdf $f_1 \equiv f$.

Hence, we can combine (8) and (21) to compute the Laplace transform of the renewal function m:

$$\hat{m}(s) \equiv \int_0^\infty e^{-st} m(t)\, dt = \sum_{n=1}^\infty \hat{F}_n(s) = \sum_{n=1}^\infty (\hat{f}_n(s)/s) = \sum_{n=1}^\infty (\hat{f}(s)^n/s) \ . \tag{22}$$

Hence,

$$\hat{m}(s) = \sum_{n=1}^\infty (\hat{f}(s)^n/s) = \frac{\hat{f}(s)}{s(1-\hat{f}(s))} = \frac{s\hat{F}(s)}{s - s^2\hat{F}(s)} \ . \tag{23}$$

As a consequence, we see that there is a *one-to-one correspondence* between cdf's F and renewal functions m. In particular, we can express $\hat{f}$ in terms of $\hat{m}$, as well as vice versa: In particular, given (23), we have the following expression going the other way:

$$s\hat{F}(s) = \hat{f}(s) = \frac{s\hat{m}(s)}{1 + s\hat{m}(s)} \ . \tag{24}$$

As another consequence of (23), we can compute any renewal function $m(t)$ by numerically inverting its Laplace transform, provided that you can evaluate the Laplace transform of F, which is $\hat{F}(s) = \hat{f}(s)/s$. That is so because we can compute the transform $\hat{m}(s)$ if we can compute the transform $\hat{f}(s)$.

We can also obtain the expression for the Laplace transform $\hat{m}$ from the renewal equation for $m(t)$,

$$m(t) = F(t) + \int_0^t m(t-y)\, dF(y) = F(t) + \int_0^t m(t-y) f(y)\, dy \ .$$

(More on the renewal equation below.) By taking Laplace transforms in this integral equation, we get

$$\hat{m}(s) = \hat{F}(s) + \hat{m}(s)\hat{f}(s) = \frac{\hat{f}(s)}{s} + \hat{m}(s)\hat{f}(s) = \hat{F}(s) + s\hat{F}(s)\hat{m}(s) \ ,$$

from which we derive the same formula above.

5.3 Convolution and Transforms

In this section we digress to provide some general technical background.

(a) definition of convolution

Consider two sequences of real numbers, $a \equiv \{a_k : k \geq 0\}$ and $b \equiv \{b_k : k \geq 0\}$. We can form a new sequence $c \equiv \{c_k : k \geq 0\}$ that is the *convolution* of the first two sequences, denoted by $c = a * b$, by letting

$$c_n = \sum_{k=0}^{k=n} a_k b_{n-k}, \quad n \geq 0 \ .$$

Similarly, consider two functions of a nonnegative real variable, $f \equiv \{f(t) : t \geq 0\}$ and $g \equiv \{g(t) : t \geq 0\}$. We can form a new function $h \equiv \{h(t) : t \geq 0\}$ that is the *convolution* of the first two functions, denoted by $h = f * g$, by letting

$$h(t) = \int_0^t f(y)g(t-y)\,dy, = \int_0^t f(t-y)g(y)\,dy, \quad t \geq 0 \ .$$

(b) associated transforms

For the sequences a, b and c, (under appropriate regularity conditions), we can form associated *generating functions*, by letting

$$\hat{a}(z) \equiv \sum_{k=0}^{\infty} z^k a_k$$

and similarly for $\hat{b}(z)$ and $\hat{c}(z)$. It is easy to see that the convolution property for the sequences a, b and c is equivalent to the transform (generating function) equation

$$\hat{c}(z) = \hat{a}(z)\hat{b}(z)$$

for all (allowed) z.

Similarly, for the functions f, g and h, (under appropriate regularity conditions), we can form associated *Laplace transforms*, by letting

$$\hat{f}(s) \equiv \int_0^{\infty} e^{-st} f(t)\,dt \ ,$$

for all complex variables s with positive real part, and similarly for $\hat{g}(s)$ and $\hat{h}(s)$. It is easy to see that the convolution property for the functions f, g and h is equivalent to the transform (Laplace transform) equation

$$\hat{h}(s) = \hat{f}(s)\hat{g}(s)$$

for all (allowed) s.

(c) probability applications

The above convolution and transform relations are general properties of sequences and functions, without probability playing a role. There are important applications to probability. However, there can be confusion if you are not careful in your treatment of pmf's, pdf's, cdf's and random variables. Almost all difficulty comes from not being careful about the definitions of these basic probability model elements.

Let X and Y be nonnegative random variables with cdf's F and G, respectively. Assume that X and Y are independent. Suppose that F and G have pdf's f and g. Let $\hat{f}_X(s)$ be the Laplace transform of X, which means

$$\hat{f}_X(s) = E[e^{-sX}] = \int_0^{\infty} e^{-sx} dF(x) = \int_0^{\infty} e^{-sx} f(x)\,dx \ .$$

Let H be the cdf and h be the pdf of $X+Y$. Then the pdf h of $X+Y$ is the convolution of the pdf's of f and g

$$h(t) = \int_0^t f(t-x)g(x)\,dx = \int_0^t f(x)g(t-x)\,dx \ .$$

If you take Laplace transforms, you get

$$\hat{h}(s) = \hat{f}(s)\hat{g}(s) \ ,$$

as above.

A similar equation holds for cdf's, but you have to be careful:

$$H(t) = \int_0^t F(t-x)dG(x) \int_0^t F(t-x)g(x)\,dx = \int_0^t F(x)g(t-x)\,dx \ .$$

When we consider Laplace transforms, you have to remember that the Laplace transform of the cdf is not the same as the Laplace transform of the pdf. (This is an important issue in renewal theory.) In particular, applying integration by parts (p. 150 of Feller II), we see that

$$\hat{F}(s) \equiv \int_0^\infty e^{-st}F(t)\,dt = \frac{\hat{f}(s)}{s} \ .$$

Then

$$\hat{H}(s) = \frac{\hat{h}(s)}{s} = \hat{F}(s)\hat{g}(s) = \frac{\hat{f}(s)}{s}\hat{g}(s) \ .$$

6 The Solution of the Renewal Equation in General

Equations (4) and (9) are special cases. In general we have the *renewal equation*

$$g(t) = h(t) + \int_0^t g(t-s)\,dF(s) = h(t) + \int_0^t g(t-s)f(s)\,ds \ , \tag{25}$$

where F is the cdf of the inter-renewal time X and h is an arbitrary function (satisfying regularity conditions). In applications, the challenge is to set up this equation properly, which means identifying the appropriate function $h(t)$ to use in (25). (Here the function h is an arbitrary function, chosen appropriately for the particular application. It is *not* intended to be the pdf of the cdf H in the previous section. The function H^c there is a special case of the function h here.)

Once we have equation (25), it suffices to solve it for g. (In equation (25) the desired function g appears on both sides; we need to find an explicit expression for g.

Equation (25) has solution

$$g(t) = h(t) + \int_0^t h(t-s)\,dm(s) = h(t) + \int_0^t h(t-s)\nu(s)\,ds \ . \tag{26}$$

It is understood that the final expression in each case depends on extra regularity conditions. As with (4) and (9), we go from (25) to (26) by replacing the two terms inside the integral: We replace g by h and we replace dF by dm.

To justify going from (25) to (26), we can take Laplace transforms. Starting from (25), we get

$$\hat{g}(s) = \hat{h}(s) + \hat{g}(s)\hat{f}(s) = \frac{\hat{h}(s)}{1-\hat{f}(s)} \tag{27}$$

On the other hand, starting from (26), and applying (7), we have

$$
\begin{aligned}
\hat{g}(s) &= \hat{h}(s) + \hat{h}(s)\hat{\nu}(s) \\
&= \hat{h}(s) + \hat{h}(s)s\hat{m}(s) \\
&= \hat{h}(s) + \hat{h}(s)\frac{\hat{f}(s)}{1-\hat{f}(s)} = \frac{\hat{h}(s)}{1-\hat{f}(s)} \ ,
\end{aligned} \tag{28}
$$

agreeing with (27).

7 The Key Renewal Theorem

The key renewal theorem provides a limit as $t \to \infty$, given the solution of the renewal equation in (26). We need the inter-renewal cdf F to be non-lattice and we need the function h to be directly Riemann integrable (DRI), an involved technical term. A sufficient condition for h to be DRI is for h to be nonnegative, non-increasing and integrable (over $[0,\infty)$), which always holds for the alternating renewal process, so we did not need the extra condition there. Given (26),

$$
g(t) \to \frac{\int_0^\infty h(x)\, dx}{E[X]} \quad \text{as} \quad t \to \infty \ .
$$

We have applied the argument in (13) above.