IEOR 3106, Fall 2013, Professor Whitt

Professor Whitt, Tuesday, November 26

Introduction to Brownian Motion, Chapter 10

1. Basic Properties: stationary and independent increments (§10.1)

a. Read Definition 10.1. Like the Poisson process, Brownian motion (BM) has **stationary and independent increments**, but BM has all real numbers as possible states as well as continuous time (in the positive halfline $[0, \infty)$. In addition, BM has **continuous sample paths**, i.e., the sample paths of BM are continuous functions of time.

b. **Standard Brownian motion**: Standard BM has parameters $\mu = 0$ and $\sigma^2 = 1$: $B(t)$ has the normal distribution $N(0, t)$, where $N(a, b)$ denotes a normally distributed random variable with mean a and variance b. Then

$$X(t) \equiv B(t; X(0), \mu, \sigma^2) = X(0) + \mu t + \sigma B(t)$$

is Brownian motion with initial position (state) $X(0)$, taken to be deterministic and known, drift rate μ and variance parameter σ^2. In particular, $X(t)$ has the **normal distribution** $N(X(0) + \mu t, \sigma^2 t)$ for each t, i.e.,

$$E[X(t)] = X(0) + \mu t \quad \text{and} \quad Var(X(t)) = \sigma^2 t.$$

The stochastic process $\{X(t) : t \geq 0\}$ is BM with starting state $X(0)$, **drift parameter** μ and **variance parameter** σ^2. As a consequence, $(X(t) - E[X(t)])/\sqrt{Var(X(t))}$ is distributed as $N(0, 1)$, so that we can apply the familiar reasoning (without using an approximation based on the central limit theorem):

$$P(X(t) > c) = P\left(\frac{X(t) - E[X(t)]}{\sqrt{Var(X(t))}} > \frac{c - E[X(t)]}{\sqrt{Var(X(t))}}\right) = P\left(N(0,1) > \frac{c - E[X(t)]}{\sqrt{Var(X(t))}}\right),$$

and compute using the table on page 82.

(c) $E[B(2)B(3)] = 2, \quad$ use: $\quad B(3) = B(2) + (B(3) - B(2))$.

(d) The probability law of a stochastic process is specified by its **finite-dimensional distributions**. We can exploit the stationary and independent increments property to write down the finite-dimensional distributions of BM. See (10.3).

(e) We can apply the finite-dimensional distributions to obtain the **conditional density**

$$f_{s|t}(x|y) \equiv f_{B(s)|B(t)}(x|y), \quad \text{pdf of cdf} \quad F_{B(s)|B(t)}(x|y) \equiv P(B(s) \leq x | B(t) = y),$$

for $0 < s < t$ (the hard case) as well as for $0 < t < s$ (the easy case). See (10.4) and display above.

2. Hitting Times and the Maximum Variable (§10.2)

(a) The **Reflection Principle**.

For $a > 0$, let T_a be the hitting time of a by BM, i.e., the first time that BM hits a.

$$P(T_a \leq t) = P(\max_{0 \leq s \leq t} B(s) > a) = 2P(B(t) > a) = 2P(N(0,t) > a) = 2P(N(0,1) > a/\sqrt{t}).$$

(b) Let T be the first hitting time of either $-a$ or $+b$ by BM B starting at 0, for $a > 0$ and $b > 0$. By martingale property (next class), $E[B(T)] = E[B(0)] = B(0) = 0$. Hence,

$$P(B(T) = b) = \frac{a}{a+b} = 1 - P(B(T) = -a).$$

(b) By working with martingales, we have $\{B(t)^2 - t : t \geq 0\}$ a martingale and

$$E[B(T)^2 - T] = E[B(0)^2 - 0] = 0,$$

so that, for the process $\{B(t) : t \geq 0\}$,

$$E[T] = E[B(T)^2 = Var(B(T)) = \frac{ab^2}{a+b} + \frac{ba^2}{a+b} = ab.$$

If we work instead with $X(t) = \sigma B(t)$, then $E[X(T)]$ is the same, but $E[T] = ab/\sigma^2$.

3. A Simple Model for a Stock Price

Problem 1 on the 2011 final exam:

1. Oatpower, Inc. (30 points)

Ever since the 2003 power outage in the northeastern United States, there has been growing investor enthusiasm for the company Oatpower, Inc., which is developing a new way to efficiently generate vast power from ordinary oats. Oatpower claims that it will be possible to generate sufficient power from a single cup of oats to run a subway train for ten years. If Oatpower is successful, subways and elevators will no longer have to depend on America's aging electric power grid. The power generation method is highly secret, but there is a rumor that it is based on a surprising chemical reaction between oats and Raspberry Snapple.

The current price of Oatpower stock is \$100 per share. Suppose that the Oatpower stock price over time (measured in years) can be modelled as the stochastic process $\{S(t) : t \geq 0\}$, where

$$S(t) \equiv 100 + 5B(t), \quad t \geq 0,$$

and $\{B(t) : t \geq 0\}$ is standard (drift zero, unit variance) Brownian motion.

(a) (4 points) Calculate $E[S(4)]$ and $E[S(4)^2]$.

(b) (4 points) Calculate $P(S(4) > 110)$.

(c) (5 points) Let T_s be the first time that the stock price reaches the level s. Calculate $P(T_{110} \leq 4)$.

(d) (5 points) Let $T \equiv \min\{T_{90}, T_{140}\}$. Calculate $E[S(T)]$ and $E[T]$.

(e) (5 points) Calculate $P(T_{90} < T_{140} < T_{80})$.

(f) (3 points) Calculate $E[S(1)|S(4) = 120]$.

(g) (4 points) Calculate $E[S(1)^2|S(4) = 120]$.

Solutions

(a) (4 points) Calculate $E[S(4)]$ and $E[S(4)^2]$.

This problem is about Chapter 10.

$$E[S(4)] = E[100 + 5B(4)] = 100 + 5E[B(4)] = 100$$

because $E[B(t)] = 0$ for all t. Since $Var(a + bX) = b^2 Var(X)$ for any random variable X,

$$Var(S(4)) = Var(5B(4)) = 25 Var(B(4)) = 25 \times 4 = 100.$$

Then the second moment is

$$E[S(4)^2] = Var(S(4)) + E[S(4)]^2 = 100 + (100)^2 = 10,100$$

(b) (4 points) Calculate $P(S(4) > 110)$.

From part (a), $S(4)$ is distributed as $N(100, 100)$. Hence,

$$\begin{aligned} P(S(4) > 110) &= P(N(100,100) > 110) = P(100 + 10N(0,1) > 110) \\ &= P(N(0,1) > 1) \approx 0.16 \end{aligned}$$

by the table on p. 82.

(c) (5 points) Let T_s be the first time that the stock price reaches the level s. Calculate $P(T_{110} \leq 4)$.

$$P(T_{110} \leq 4) = P(\max_{0 \leq t \leq 4} \{S(t)\} > 110) = 2P(S(4) > 110) = 2(0.16) = 0.32$$

by §10.2 of the book and then by part (a).

(d) (5 points) Let $T \equiv \min\{T_{90}, T_{140}\}$. Calculate $E[S(T)]$ and $E[T]$.

It is helpful to rephrase the question in terms of ordinary Brownian motion. The hitting time T is distributed the same as $T \equiv \min\{T_{-2}, T_8\}$ for ordinary Brownian motion. That is, the original stochastic process $S(t)$ hits either 90 or 140 the same time that the component $B(t)$ hits either -2 or $+8$.

We use the optional stopping theorem with martingales to obtain

$$E[B(T)] = E[B(0)] = 0, \quad \text{so that} \quad E[S(T)] = E[S(0)] = 100.$$

For the second part we can However, by Exercise 10.18 and by the lecture notes, $B(t)^2 - t$ is a martingale so that

$$E[T] = 2 \times 8 = 16$$

(e) (5 points) Calculate $P(T_{90} < T_{140} < T_{80})$.

This is just like Exercise 10.5. First, since we start at $S(0) = 100$, we use $E[S(T)] = 0$ to obtain

$$P(T_{90} < T_{140}) = \frac{4}{4+1} = \frac{4}{5}.$$

After we hit 90, we have a second independent problem of hitting 140 before 80.

$$P(T_{140} < T_{80} | T_{90} < T_{140}) = \frac{1}{1+5} = \frac{1}{6}.$$

Since these two events are independent,

$$P(T_{90} < T_{140} < T_{80}) = \left(\frac{4}{5}\right) \times \left(\frac{1}{6}\right) = \frac{2}{15}$$

(f) (3 points) Calculate $E[S(1)|S(4) = 120]$.

Here you should use the first equation in display (10.4) in §10.1.

$$E[S(1)|S(4) = 120] = 100 + \frac{1}{4}20 = 105$$

Again, it may be helpful to rephrase the question in terms of ordinary Brownian motion.

$$E[S(1)|S(4) = 120] = 100 + 5E[B(1)|B(4) = 4] = 100 + 5 \times 1 = 105.$$

(g) (4 points) Calculate $E[S(1)^2|S(4) = 120]$.

Here we should use the first equation in display (10.4) in §10.1.

$$Var(S(1)|S(4) = 120) = \left(\frac{1 \times 3}{4}\right) 25 = \frac{75}{4}.$$

Again, it may be helpful to rephrase the question in terms of ordinary Brownian motion.

$$Var(S(1)|S(4) = 120) = 25Var(B(1)|B(4) = 4) = 25\left(\frac{1 \times 3}{4}\right) = \frac{75}{4}.$$

Hence, the second moment is

$$E[S(1)^2|S(4) = 120] = Var(S(1)|S(4) = 120) + (E[S(1)|S(4) = 120])^2 = \frac{75}{4} + (105)^2 = 11,043.75$$

It would be OK to omit the final calculation.
