

IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2013, Professor Whitt, Tuesday, December 3

Brownian Motion, Gambler's Ruin and Martingales

Sections 10.1-10.3. Brownian motion

We here discuss the gambler's ruin problem for Brownian motion, using martingales.

1. Basic Definitions

Let $B(t) : t \geq 0$ be **standard** ($\mu = 0$, zero-drift, and $\sigma^2 = 1$, unit variance) **Brownian motion** (BM).

A stochastic process $\{Y(t) : t \geq 0\}$ is a **martingale** (MG) with respect to another stochastic process $\{Z(t) : t \geq 0\}$ if

$$E[Y(t)|Z(u), 0 \leq u \leq s] = Y(s) \quad \text{for } 0 < s < t .$$

As an extra technical regularity condition, we require that $E[|Y(t)|] < \infty$ for all t as well.

The stochastic process $\{Z(t) : t \geq 0\}$ above is giving relevant information. The segment $\{Z(s) : 0 \leq s \leq t\}$ gives the **history** up to time t . Often the information process Z is just the given stochastic process Y . Then we just say that $\{Y(t) : t \geq 0\}$ is a **martingale** (MG), without saying "with respect to." We will use the fact that standard BM $\{B(t) : t \geq 0\}$ is a martingale with respect to itself. Then we just say that standard BM is a martingale. But we shall be interested in martingales with respect to BM that are themselves appropriate functions of BM. It can be shown that they too are simply martingales (with respect to their own "internal" history), but we will simply show that they are martingales with respect to BM..

A nonnegative random variable T is a **stopping time** relative to a stochastic process $\{Z(t) : t \geq 0\}$ if, for any time t , the event $\{T \leq t\}$ depends on $Z(s)$ only for $0 \leq s \leq t$. Stopping before time t depends only upon the history up to time t . The event that a stopping time T is less than or equal to t cannot depend on the future of the reference stochastic process $\{Z(s) : s \geq 0\}$ after time t .

The **optional stopping theorem** says that, under regularity conditions, when Y is a martingale with respect to Z and T is a stopping time relative to Z , that

$$E[Y(T)] = E[Y(0)] .$$

We will apply that below. That also can be expressed by saying that $\{Y(0), Y(T)\}$ is a MG. We remark that the extra regularity conditions are important, because without them the result need not be true, but they will be satisfied in the cases we consider.

2. The Gambler's Ruin Problem

Let $X(t) \equiv \sigma B(t) + \mu t$, for $t \geq 0$. Then X is Brownian motion with drift μ and diffusion or variance coefficient σ^2 . Then $X(t) \stackrel{d}{=} N(\mu t, \sigma^2 t)$ for each $t \geq 0$. Suppose that we start with an initial amount of money w and gamble according to the stochastic process X , i.e., with Brownian motion with drift. Thus our wealth at time t is given by

$$W(t) = w + X(t) = w + \sigma B(t) + \mu t, \quad t \geq 0 .$$

We now ask what is the probability we win x (reach wealth $w + x$) before we lose our initial wealth w (reach wealth 0). Let T be the first time that we either win x or lose w , i.e.,

$$T \equiv \inf \{t \geq 0 : W(t) = 0 \text{ or } w + x\} = \inf \{t \geq 0 : X(t) = -w \text{ or } +x\} .$$

We want to find the probability that we win, i.e., reach wealth $w + x$ before being ruined (reaching wealth 0):

$$p \equiv P(W(T) = w + x) = P(X(T) = x)$$

and we want to find the expected time before the “game” ends, i.e., $E[T]$. There are two cases: (i) no drift ($\mu = 0$), and (ii) drift ($\mu \neq 0$). In both cases we can use martingales.

3. Three Basic Martingales

We now define three basic martingales with respect to standard Brownian motion B . The **linear martingale** is Brownian motion itself $\{B(t) : t \geq 0\}$. The **quadratic martingale** is $\{B(t)^2 - t : t \geq 0\}$. The **exponential martingale** is $\{e^{\theta B(t) - \theta^2 t/2} : t \geq 0\}$. We find these last two by considering $E[B(t)^2]$ and $E[e^{\theta B(t)}]$ and seeing what deterministic adjustment guarantees that the expectation at t equals the expectation at 0. That does not prove the MG property, but that is the first step. The MG property follows from the stationary and independent increments of BM.

For example, we now show that the quadratic martingale $\{B(t)^2 - t : t \geq 0\}$ is indeed a martingale with respect to Brownian motion. We exploited the independent increments property, writing

$$\begin{aligned} E[B(t+u)^2 - (t+u)|B(s), \quad 0 \leq s \leq t] &= E[B(t+u)^2|B(s), \quad 0 \leq s \leq t] - (t+u) \\ &= E[(B(t) + B(t+u) - B(t))^2|B(s), \quad 0 \leq s \leq t] - (t+u) \\ &= E[B(t)^2 + 2B(t)(B(t+u) - B(t)) \\ &\quad + (B(t+u) - B(t))^2|B(s), \quad 0 \leq s \leq t] - (t+u) \\ &= B(t)^2 + 2B(t)E[B(t+u) - B(t)|B(s), \quad 0 \leq s \leq t] \\ &\quad + E[(B(t+u) - B(t))^2|B(s), \quad 0 \leq s \leq t] - (t+u) \\ &= B(t)^2 + E[(B(t+u) - B(t))^2|B(s), \quad 0 \leq s \leq t] - (t+u) \\ &= B(t)^2 + E[(B(t+u) - B(t))^2] - (t+u) \\ &= B(t)^2 + E[(B(u) - B(0))^2] - (t+u) \\ &= B(t)^2 + E[B(u)^2] - (t+u) \\ &= B(t)^2 + Var[B(u)] - (t+u) \\ &= B(t)^2 + u - (t+u) \\ &= B(t)^2 - t , \end{aligned}$$

as claimed.

4. Three Basic Martingales Applied to the Gambler’s Ruin Problem

The three basic martingales can be applied with the optional stopping theorem to obtain the desired quantities p and $E[T]$ in the gambler’s ruin problem, with and without drift. The appropriate martingale in each case is given in Table 1.

desired quantity	no drift	drift
p	linear martingale	exponential martingale
$E[T]$	quadratic martingale	linear martingale

Table 1: The martingales used to treat the gambler's ruin problem for Brownian motion, with and without drift.

5. The Case without Drift

The case without drift is easier. For the Gambler's ruin problem expressed in terms of x and w , we use the linear MG to get

$$p = \frac{w}{x+w}$$

and then we use the quadratic MG to get

$$E[T] = \frac{xw}{\sigma^2} .$$

We now explain: We first consider standard Brownian motion, which starts at $B(0) = 0$. For the first step, we let T be the first time that standard Brownian motion first hits either $+a$ or $-b$. We apply the optional sampling theorem to get

$$E[B(T)] = E[B(0)] = 0 ,$$

but $B(T)$ necessarily is either $+a$ or $-b$. Let p be the probability that $B(T) = a$. Then the equation is

$$E[B(T)] = pa + (1-p)(-b) = 0 ,$$

but that implies that

$$p = \frac{b}{a+b} ,$$

which reduces to the formula above when we use $X(t) = \sigma B(t)$, $a = x$ and $b = w$.

By a minor variant of the same reasoning, but with the quadratic martingale, we get

$$E[B(T)^2 - T] = E[B(0)^2 - 0] = 0 ,$$

which implies that

$$E[T] = E[B(T)^2] = pa^2 + (1-p)b^2 = \frac{ab(a+b)}{a+b} = ab .$$

If we instead consider the stochastic process $X(t) = \sigma B(t)$, then we see that the appropriate quadratic martingale is instead $\{X(t)^2 - \sigma^2 t : t \geq 0\}$. We then get

$$E[X(T)^2 - \sigma^2 T] = E[B(0)^2 - \sigma^2 0] = 0 ,$$

which implies that

$$E[\sigma^2 T] = \sigma^2 E[T] = E[X(T)^2] = pa^2 + (1-p)b^2 = \frac{ab(a+b)}{a+b} = ab$$

and

$$E[T] = \frac{ab}{\sigma^2} .$$

The final two sections below are optional.

6. The Case with Drift

Start by applying the exponential martingale associated with standard Brownian motion $B(t)$ to find p . The first step is to show that the exponential martingale $\{e^{\theta B(t) - \theta^2 t/2} : t \geq 0\}$ is in fact a martingale. Then we apply the Martingale Stopping Theorem (or the optional stopping theorem) to get

$$E[e^{\theta B(T) - \theta^2 T/2}] = E[e^{\theta B(0) - \theta^2 0/2}] = e^0 = 1 .$$

Thus, upon expressing $B(t)$ in terms of $X(t)$, we have

$$E[e^{\theta[(X(T) - \mu T)/\sigma] - \theta^2 T/2}] = E[e^{(\theta X(T)/\sigma) - (\theta \mu T/\sigma) - \theta^2 T/2}] = 1 .$$

Now let

$$\theta = \frac{-2\mu}{\sigma} .$$

That serves to knock out the last two terms in the exponent, giving

$$E[e^{(-2\mu X(T)/\sigma^2)}] = 1 .$$

But then use the fact that $X(T) = x$ with probability p and $X(T) = -w$ with probability $1 - p$. Thus we get

$$pe^{-2\mu x/\sigma^2} + (1 - p)e^{2\mu w/\sigma^2} = 1$$

or

$$p = \frac{e^{+2\mu w/\sigma^2} - 1}{e^{+2\mu w/\sigma^2} - e^{-2\mu x/\sigma^2}} .$$

That is the natural way to write the formula when $\mu > 0$; then both the numerator and denominator are positive. You can check that $0 < p < 1$.

If instead $\mu < 0$, then we would write

$$p = \frac{1 - e^{+2\mu w/\sigma^2}}{e^{-2\mu x/\sigma^2} - e^{+2\mu w/\sigma^2}} .$$

That makes the numerator and denominator positive.

Now, knowing p , we can apply the linear martingale to find $E[T]$. we have

$$E[B(T)] = 0$$

or, equivalently,

$$E[(X(T) - \mu T)/\sigma] = 0$$

or

$$E[T] = \frac{E[X(T)]}{\mu} = \frac{(px - (1 - p)w)}{\mu} .$$

We then use the known p and solve for $E[T]$. We get a ratio of linear combinations of exponential terms.

7. Other Brownian Martingales

There is also the **cubic martingale**

$$B(t)^3 - 3tB(t), \quad t \geq 0,$$

and the **quartic martingale**

$$B(t)^4 - 6tB(t)^2 + 3t^2, \quad t \geq 0.$$

For more discussion, see Exercises 10.16-10.24 in Ross and the online lecture notes on martingales. Also see exercises 7.13-7.15. See p. 382 in his more advanced book, *Stochastic Processes* for the case with drift above.