IEOR 3106, Fall 2013, Professor Whitt

Professor Whitt, Thursday, December 5

Old Brownian Exam Problems

1. Oatpower, Inc. (2011 final exam)

Ever since the 2003 power outage in the northeastern United States, there has been growing investor enthusiasm for the company Oatpower, Inc., which is developing a new way to efficiently generate vast power from ordinary oats. Oatpower claims that it will be possible to generate sufficient power from a single cup of oats to run a subway train for ten years. If Oatpower is successful, subways and elevators will no longer have to depend on America's aging electric power grid. The power generation method is highly secret, but there is a rumor that it is based on a surprising chemical reaction between oats and Raspberry Snapple.

The current price of Oatpower stock is $100 per share. Suppose that the Oatpower stock price over time (measured in years) can be modelled as the stochastic process $\{S(t) : t \geq 0\}$, where

$$S(t) \equiv 100 + 5B(t), \quad t \geq 0,$$

and $\{B(t) : t \geq 0\}$ is standard (drift zero, unit variance) Brownian motion.

(a) (4 points) Calculate $E[S(4)]$ and $E[S(4)^2]$.

(b) (4 points) Calculate $P(S(4) > 110)$.

(c) (5 points) Let T_s be the first time that the stock price reaches the level s. Calculate $P(T_{110} \leq 4)$.

(d) (5 points) Let $T \equiv \min\{T_{90}, T_{140}\}$. Calculate $E[S(T)]$ and $E[T]$.

(e) (5 points) Calculate $P(T_{90} < T_{140} < T_{80})$.

(f) (3 points) Calculate $E[S(1)|S(4) = 120]$.

(g) (4 points) Calculate $E[S(1)^2|S(4) = 120]$.

Solutions

(a) (4 points) Calculate $E[S(4)]$ and $E[S(4)^2]$.

This problem is about Chapter 10.

$$E[S(4)] = E[100 + 5B(4)] = 100 + 5E[B(4)] = 100$$

because $E[B(t)] = 0$ for all t. Since $Var(a + bX) = b^2 Var(X)$ for any random variable X,

$$Var(S(4)) = Var(5B(4)) = 25Var(B(4)) = 25 \times 4 = 100.$$

Then the second moment is

$$E[S(4)^2] = Var(S(4)) + E[S(4)]^2 = 100 + (100)^2 = 10,100$$

(b) (4 points) Calculate $P(S(4) > 110)$.

From part (a), $S(4)$ is distributed as $N(100, 100)$. Hence,

$$\begin{aligned} P(S(4) > 110) &= P(N(100,100) > 110) = P(100 + 10N(0,1) > 110) \\ &= P(N(0,1) > 1) \approx 0.16 \end{aligned}$$

by the table on p. 82.

(c) (5 points) Let T_s be the first time that the stock price reaches the level s. Calculate $P(T_{110} \leq 4)$.

$$P(T_{110} \leq 4) = P(\max_{0 \leq t \leq 4} \{S(t)\} > 110) = 2P(S(4) > 110) = 2(0.16) = 0.32$$

by §10.2 of the book and then by part (a).

(d) (5 points) Let $T \equiv \min\{T_{90}, T_{140}\}$. Calculate $E[S(T)]$ and $E[T]$.

It is helpful to rephrase the question in terms of ordinary Brownian motion. The hitting time T is distributed the same as $T \equiv \min\{T_{-2}, T_8\}$ for ordinary Brownian motion. That is, the original stochastic process $S(t)$ hits either 90 or 140 the same time that the component $B(t)$ hits either -2 or $+8$.

We use the optional stopping theorem with martingales to obtain

$$E[B(T)] = E[B(0)] = 0, \quad \text{so that} \quad E[S(T)] = E[S(0)] = 100.$$

For the second part we can However, by Exercise 10.18 and by the lecture notes, $B(t)^2 - t$ is a martingale so that

$$E[T] = 2 \times 8 = 16$$

(e) (5 points) Calculate $P(T_{90} < T_{140} < T_{80})$.

This is just like Exercise 10.5. First, since we start at $S(0) = 100$, we use $E[S(T)] = 0$ to obtain

$$P(T_{90} < T_{140}) = \frac{4}{4+1} = \frac{4}{5}.$$

After we hit 90, we have a second independent problem of hitting 140 before 80.

$$P(T_{140} < T_{80} | T_{90} < T_{140}) = \frac{1}{1+5} = \frac{1}{6}.$$

Since these two events are independent,

$$P(T_{90} < T_{140} < T_{80}) = \left(\frac{4}{5}\right) \times \left(\frac{1}{6}\right) = \frac{2}{15}$$

(f) (3 points) Calculate $E[S(1)|S(4) = 120]$.

Here you should use the first equation in display (10.4) in §10.1.

$$E[S(1)|S(4) = 120] = 100 + \frac{1}{4}20 = 105$$

Again, it may be helpful to rephrase the question in terms of ordinary Brownian motion.

$$E[S(1)|S(4) = 120] = 100 + 5E[B(1)|B(4) = 4] = 100 + 5 \times 1 = 105.$$

(g) (4 points) Calculate $E[S(1)^2|S(4) = 120]$.

Here we should use the first equation in display (10.4) in §10.1.

$$Var(S(1)|S(4) = 120) = \left(\frac{1 \times 3}{4}\right) 25 = \frac{75}{4}.$$

Again, it may be helpful to rephrase the question in terms of ordinary Brownian motion.

$$Var(S(1)|S(4) = 120) = 25Var(B(1)|B(4) = 4) = 25\left(\frac{1 \times 3}{4}\right) = \frac{75}{4}.$$

Hence, the second moment is

$$E[S(1)^2|S(4) = 120] = Var(S(1)|S(4) = 120) + (E[S(1)|S(4) = 120])^2 = \frac{75}{4} + (105)^2 = 11,043.75$$

It would be OK to omit the final calculation.

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2. Problem 4 from 2012 Exam

Ten Independent Stocks: Two Investment Strategies (25 points +4 bonus)

You have decided to invest \$800 by buying ten shares at time 0 of stocks initially priced at \$80 per share. Assume that the 10 different stock prices evolve independently over time, with the price of stock j evolving according to the model

$$S_j(t) \equiv 80 + 2.5t + 5B_j(t), \quad t \geq 0,$$

where $\{B_j(t) : t \geq 0\}$ is a standard (drift zero, unit variance) Brownian motion (BM) for each j, with the ten different BM's being stochastically independent.

(a) (2 points) Suppose that you employ a **focused investment strategy** and buy 10 shares of stock 1 at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[10S_1(4)]$ and $Var(10S_1(4))$?

(b) (2 points) With the focused investment strategy in part (a), what is the probability that you will have made a profit? That is, what is the probability that $P(10S_1(4) > 800)$?

(c) (2 points) Suppose that, instead, you decide to employ a **diversified investment strategy** and buy 1 share of each of the 10 different stocks at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[S_1(4) + \cdots + S_{10}(4)]$ and $Var(S_1(4) + \cdots + S_{10}(4))$?

(d) (2 points) With the alternative investment scheme in part (c), what is the probability that you will have made a profit? That is, what is the probability that $P(S_1(4) + \cdots + S_{10}(4) > 800)$?

(e) (5 points) True or false: Indicate whether each of the following five statements is true or false, explaining briefly:

(i) For each of the two investment strategies, the total value of the stock at time $t = 4$ is a random variable with a normal probability distribution.

(ii) The total value of the stock at time $t = 4$ has a probability distribution that is the same for both investment strategies.

(iii) An investor whose sole goal is to maximize his expected return should strongly prefer the diversified investment strategy.

(iv) An investor who wants to achieve the expected return of the focused strategy but minimize his risk, as defined by the probability of suffering a loss over the investment period $[0, 4]$, should strongly prefer the diversified investment strategy.

(v) An investor who wants to maximize the probability that he achieves at least 20% more than the expected value should strongly prefer the focused investment strategy.

(f) (4 points) Let T be the first time that the share price of stock 1 either exceeds its expected value by \$20 or falls below its expected value by \$10; i.e., let

$$T \equiv \inf \{t > 0 : S_1(t) - E[S_1(t)] \geq 20 \quad \text{or} \quad S_1(t) - E[S_1(t)] \leq -10\}.$$

What are $E[S_1(T)]$, $P(S_1(T) - E[S_1(T)] = 20)$ and $E[T]$? Briefly explain.

Problem 4 continued: dependence (in one stock over time and between stocks)

(g) (4 points) What are $E[S_1(3)|S_1(4) = 120]$ and $Var[S_1(3)|S_1(4) = 120]$?

(h) (4 points) What are the expected values: $E[S_1(3)S_1(4)]$ and $E[S_1(4)S_2(4)]$? (The stock price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ are still assumed to be independent in this part.)

(i) (BONUS [harder] 4 points) Suppose that there are stocks for which the stock prices are in fact **stochastically dependent** in various ways. If you could **control the dependence** between the stocks without altering the probability law of each individual stock price process (e.g., by picking the stocks in some clever way), then how could you do better than the diversified investment strategy above? That is, how could you (i) control the dependence and (ii) make an investment strategy in order to reduce the risk (the probability of not making a profit) while leaving the overall expected value of the investment at time 4 (or any other time) unchanged?

SOLUTIONS

(a) (2 points) Suppose that you employ a **focused investment strategy** and buy 10 shares of stock 1 at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[10S_1(4)]$ and $Var(10S_1(4))$?

$$
\begin{aligned}
E[10S_1(4)] &= 10E[80 + (2.5 \times 4) + 5B_1(4)] = 10(80 + 10 + 5E[B_1(4)] \\
&= 10(80 + 10 + (5 \times 0)] = 10(90) = 900
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
Var[10S_1(4)] &= (10)^2 Var(80 + (2.5 \times 4) + 5B_1(4)) = 100 Var(5B_1(4)) \\
&= 100 \times 25 \times Var(B(4)) = 100 \times 25 \times 4 = 10,000
\end{aligned} \tag{2}
$$

(b) (2 points) With the focused investment strategy in part (a), what is the probability that you will have made a profit? That is, what is the probability that $P(10S_1(4) > 800)$?

By part (a), $10S_1(4) \stackrel{\text{d}}{=} N(900, 10^4)$, i.e. $10S_1(4)$ is normally distributed with the mean and variance above. So that

$$
\begin{aligned}
P(10S_1(4) > 800) &= P\left(\frac{10S_1(4) - E[10S_1(4)]}{\sqrt{Var(10S_1(4))}} > \frac{800 - E[10S_1(4)]}{\sqrt{Var(10S_1(4))}}\right) \\
&= P\left(N(0,1) > \frac{800 - 900}{100}\right) = P\left(N(0,1) > -1\right) \\
&= 1 - P\left(N(0,1) \le 1\right) \approx 1 - 0.16 = 0.84
\end{aligned} \tag{3}
$$

(c) (2 points) Suppose that, instead, you decide to employ a **diversified investment strategy** and buy 1 share of each of the 10 different stocks at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[S_1(4) + \cdots + S_{10}(4)]$ and $Var(S_1(4) + \cdots + S_{10}(4))$?

The mean is the same as in part (a), but the variance is less. The variance of the sum of independent random variables is the sum of the variances, whereas $Var(cX) = c^2 Var(X)$. In particular,

$$
\begin{aligned}
E[S_1(4) + \cdots + S_{10}(4)] &= 10E[S_1(4)] = 10(80 + 10 + 5E[B_1(4)] \\
&= 10(80 + 10 + (5 \times 0)] = 10(90) = 900
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
Var[S_1(4) + \cdots + S_{10}(4)] &= 10 Var(80 + (2.5 \times 4) + 5B_1(4)) = 10 Var(5B_1(4)) \\
&= 10 \times 25 \times Var(B(4)) = 10 \times 25 \times 4 = 1,000
\end{aligned} \tag{5}
$$

So the variance is 10 tens smaller than in part (a)!

(d) (2 points) With the alternative investment scheme in part (c), what is the probability that you will have made a profit? That is, what is the probability that $P(S_1(4)+\cdots+S_{10}(4) > 800)$?

Since the variance is $1,000$ instead of $10,000$, the standard deviation is about $\sqrt{1000} \approx 31$ instead of 100.

By part (c), $S_1(4) + \cdots + S_{10}(4) \stackrel{\mathrm{d}}{=} N(900, 10^3)$, i.e. $S_1(4) + \cdots + S_{10}(4)$ is normally distributed with the mean and variance above. So that

$$\begin{aligned}
& P(S_1(4) + \cdots + S_{10}(4) > 800) \\
& \quad = P\left(\frac{S_1(4) + \cdots + S_{10}(4) - E[S_1(4) + \cdots + S_{10}(4)]}{\sqrt{S_1(4) + \cdots + S_{10}(4)}} > \frac{800) - ES_1(4) + \cdots + S_{10}(4)]}{\sqrt{Var(S_1(4) + \cdots + S_{10}(4))}}\right) \\
= \; & P\left(N(0,1) > \frac{800 - 900}{31}\right) \approx P\left(N(0,1) > -3.1\right) \\
= \; & 1 - P\left(N(0,1) \leq 3.1\right) \approx 1 - 0.001 = 0.999 \qquad (6)
\end{aligned}$$

(e) (5 points) True or false: Indicate whether each of the following five statements is true or false, explaining briefly:

(i) For each of the two investment strategies, the total value of the stock at time $t = 4$ is a random variable with a normal probability distribution.

(ii) The total value of the stock at time $t = 4$ has a probability distribution that is the same for both investment strategies.

(iii) An investor whose sole goal is to maximize his expected return should strongly prefer the diversified investment strategy.

(iv) An investor who wants to achieve the expected return of the focused strategy but minimize his risk, as defined by the probability of suffering a loss over the investment period $[0, 4]$, should strongly prefer the diversified investment strategy.

(v) An investor who wants to maximize the probability that he achieves at least 20% more than the expected value should strongly prefer the focused investment strategy.

The goal here is to consolidate the knowledge gained from the previous detailed parts. For the most part, these questions can be answered given that the previous parts have been done correctly. Even if you made numerical mistakes, you could understand the main idea.

(i) TRUE, both distributions are normal.

(ii) FALSE, the variances are very different.

(iii) FALSE, the expected returns are the same with the two strategies.

(iv) TRUE, as shown by specific calculations above.

(v) TRUE, because the greater variance with the focused strategy increases the likelihood of exceeding a value above the mean. This can be shown by calculations just like those of parts

(b) and (d) above. Since 20% of 900 is 180, we compare $P(10S_1(4) > 900+180) = P(10S_1(4) > 1080)$ to $P(S_1(4) + \cdots + S_{10}(4) > 1080)$. By obvious modifications of the calculations above, we see that

$$P(10S_1(4) > 1080) > P(S_1(4) + \cdots + S_{10}(4) > 1080).$$

(f) (4 points) Let T be the first time that the share price of stock 1 either exceeds its expected value by \$20 or falls below its expected value by \$10; i.e., let

$$T \equiv \inf \{t > 0 : S_1(t) - E[S_1(t)] \geq 20 \quad \text{or} \quad S_1(t) - E[S_1(t)] \leq -10\}.$$

What are $E[S_1(T)]$, $P(S_1(T) - E[S_1(T)] = 20)$ and $E[T]$? Briefly explain.

The idea here is to apply martingales and the optional stopping theorem (OST). The stochastic process $\{S_1(t) : t \geq 0\}$ is *not* a martingale, because of the drift term, but the stochastic process $\{S_1(t) - E[S_1(t) : t \geq 0\}$ is a martingale. Indeed, when we subtract the mean, the problem is the same as if we had scaled Brownian motion itself; note that

$$S_1(t) - E[S_1(t)] = 5B_1(t), \quad t \geq 0.$$

Thus, we can apply the OST to get

$$E[S_1(T) - E[S_1(T)]] = E[S_1(0) - E[S_1(0)]] = 0,$$

so that

$$P(S_1(T) - E[S_1(T)] = 20) = 1 - P(S_1(T) - E[S_1(T)] = -10) = \frac{10}{10 + 20} = \frac{1}{3}.$$

Similarly, $\{(S_1(t) - E[S_1(t))^2 - t : t \geq 0\}$ is a martingale. Thus,

$$E[T] = \frac{10 \times 20}{Var(S(1))} = \frac{10 \times 20}{25} = 8$$

Alternatively, we can reason from the fact that $\{B(t)^2 - t : t \geq 0\}$ is a martingale.

Since $\{S_1(t) - E[S_1(t) : t \geq 0\}$ is a martingale, by the OST

$$E[S_1(T) - 80 - 2.5T] = E[S_1(T)] - 80 - 2.5E[T] = 0,$$

so that

$$E[S_1(T)] = 80 + 2.5E[T] = 100.$$

Problem 4 continued: dependence (in one stock over time and between stocks)

(g) (4 points) What are $E[S_1(3)|S_1(4) = 120]$ and $Var[S_1(3)|S_1(4) = 120]$?

This is essentially homework exercise 10.2, assigned in the last homework. We use the formulas in (10.4) of the book, given in §2 of the formula sheet for BM.

This is the conditional expectation and variance looking backwards, i.e., at time s given the value at time t for $0 < s < t$. First, substitute in and simplify

$$\begin{aligned} E[S_1(3)|S_1(4) = 120] &= E[80 + 2.5(3) + 5B(3)|80 + 2.5(4) + 5B(4) = 120] \\ &= 87.5 + 5E[B(3)|5B(4) = 30] = 87.5 + 5E[B(3)|B(4) = 6] \\ &= 87.5 + 5(4.5) = 87.5 + 22.5 = 110 \end{aligned} \tag{7}$$

For this part, you could reason directly, getting

$$E[S_1(3)|S_1(4) = 120] = 80 + (3/4)(120 - 80) = 80 + 30 = 110.$$

We now turn to the conditional variance.

$$\begin{aligned} Var[S_1(3)|S_1(4) = 120] &= Var[80 + 2.5(3) + 5B(3)|80 + 2.5(4) + 5B(4) = 120] \\ &= Var(5E[B(3)|5B(4) = 30] = 25Var[B(3)|B(4) = 6] \\ &= \frac{25 \times 3 \times (4 - 3)}{4} = \frac{75}{4} = 18.75 \end{aligned} \tag{8}$$

(h) (4 points) What are the expected values: $E[S_1(3)S_1(4)]$ and $E[S_1(4)S_2(4)]$? (The stock price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ are still assumed to be independent in this part.)

For parts (f) and (g), we reduce the analysis to basic facts involving BM $B(t)$. First, recall that, for $0 < s < t$,

$$E[B(s)B(t)] = E[B(s)(B(s) + B(t) - B(s))] = E[B(s)^2] = Var(B(s)) = s.$$

Then observe that

$$\begin{aligned} E[S_1(3)S_1(4)] &= E[(80 + 2.5(3) + 5B(3))(80 + 2.5(4) + 5B(4))] \\ &= E[(87.5 + 5B(3))(90 + 5B(4))] \\ &= (87.5 \times 90) + 450E[B(3)] + 437.5E[B(4)] + 25E[B(3)B(4)] \\ &= (87.5 \times 90) + 0 + 0 + (25 \times (3) = (87.5 \times 90) + 75 \\ &= 7875 + 75 = 7950 \end{aligned} \tag{9}$$

(The last multiplication and final calculation in the final line is not needed.)

Turning to the second question, Since the different stock prices are i.i.d., we have

$$E[S_1(4)S_2(4)] = E[S_1(4)]E[S_2(4)] = (E[S_1(4)])^2 = (90)^2 = 8100.$$

(i) (BONUS 4 points) Suppose that there are stocks for which the stock prices are in fact **stochastically dependent** in various ways. If you could **control the dependence** between the stocks without altering the probability law of each individual stock price process (e.g., by picking the stocks in some clever way), then how could you do better than the diversified investment strategy above? That is, how could you (i) control the dependence and (ii) make an investment strategy in order to reduce the risk (the probability of not making a profit) while leaving the overall expected value of the investment at time 4 (or any other time) unchanged?

The point of this part of the problem is to introduce the idea of **hedging** in finance. (For background, look at "hedge in finance" in Wikipedia.) The idea is to invest in two stocks that are strongly negatively correlated, so that excursions of one above or below the expected value will be matched by excursions in the other in the opposite direction. We look for stocks that have similar expected value behavior, but are as negatively correlated as possible. That is, we want to invest in two stocks that have -1 correlation. Recall that the **correlation** of two random variables is defined as

$$\rho_{X,Y} \equiv Corr(X,Y) \equiv \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}},$$

where

$$Cov(X,Y) \equiv E[(X - E[X])(Y - E[Y]) = E[XY] - E[X]E[Y]$$

is the **covariance**. The correlation can assume values in the interval $[-1,1]$ and only there. Thus a correlation of -1 is the most negative. That is achieved when $Y = -X$. That is what we are aiming for in the variability part of the stocks. But we want to leave the positive drift unchanged.

The stochastic dependence between two stock price processes $\{S_j(t) : t \geq 0\}$ for $j = 1, 2$ defined in terms of Brownian motions $\{B_j(t) : t \geq 0\}$ for $j = 1, 2$, as in this problem, will have dependence determined by the dependence in the underlying Brownian motions. We use this structure to artificially **control the dependence**. We can achieve perfectly negative dependence (the best possible) by letting

$$B_2(t) = -B_1(t), \quad t \geq 0.$$

Notice that the stochastic process $\{-B_1(t) : t \geq 0\}$ has the same Brownian motion probability law as a stochastic process as the original Brownian motion $\{B_1(t) : t \geq 0\}$, but of course

$$B_1(t) + B_2(t) = 0 \quad \text{for all} \quad t \geq 0.$$

As before, for $j = 1, 2$, we let

$$S_j(t) \equiv 80 + 2.5t + 5B_j(t), \quad t \geq 0,$$

but where $B_2(t) \equiv -B_1(t)$, $t \geq 0$, as above.

Our new **hedging strategy based on dependence** involves buying equal amounts of the two stocks. We would thus buy 5 shares of stock $\{S_1(t) : t \geq 0\}$ and 5 shares of $\{S_2(t) : t \geq 0\}$. Since the Brownian motions have the proper probability distribution, these processes individually have the correct probability law. However, we achieve our objective, because

$$5S_1(t) + 5S_2(t) = 800 + 25t, \quad t \geq 0,$$

without any variability at all. Note that

$$E[5S_1(t) + 5S_2(t)] = 800 + 25t \quad \text{and} \quad Var(5S_1(t) + 5S_2(t)) = 0 \quad \text{for all} \quad t \geq 0.$$

so that, for $t = 4$,

$$E[5S_1(4) + 5S_2(4)] = 900 \quad \text{and} \quad Var(5S_1(4) + 5S_2(4)) = 0 \quad \text{for all} \quad t \geq 0.$$

We have kept the same mean, but reduced the variance to 0.

In practice, something like this is achieved with financial instruments that are very negatively correlated. Usually some of the expected value must be sacrificed in order to successfully hedge.