IEOR 3106: Introduction to Operations Research: Stochastic Models

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Supplementary Notes on Martingales and Brownian Motion

1. Martingales

We start by defining a martingale, working in discrete time.

Definition 0.1 *Let $\{X_n : n \geq 0\}$ and $\{Y_n : n \geq 0\}$ be stochastic processes (sequences of random variables). We say that $\{X_n : n \geq 0\}$ is a **martingale** with respect to $\{Y_n : n \geq 0\}$ if*

$$(i) E[|X_n|] < \infty \quad \textit{for all} \quad n \geq 0$$

and

$$(ii) E[X_{n+1}|Y_0, Y_1, \ldots, Y_n] = X_n \quad \textit{for all} \quad n \geq 0 \ .$$

a. More on Definition 0.1.

In Definition 0.1 we think of the stochastic process $\{Y_n : n \geq 0\}$ constituting **the history or information**. Then $\{Y_k : 0 \leq k \leq n\}$ is the history up to (and including) time n. The random variables Y_k could be random vectors, as we illustrate below. We simply say that $\{X_n : n \geq 0\}$ is a martingale if $\{X_n : n \geq 0\}$ is a martingale with respect to $\{X_n : n \geq 0\}$; i.e., if the history process $\{Y_n : n \geq 0\}$ is the stochastic process $\{X_n : n \geq 0\}$ itself. We then also say that $\{X_n : n \geq 0\}$ is a martingale with respect to its internal history (the history generated by $\{X_n : n \geq 0\}$).

In the literature on martingales, the histories are usually characterized via sigma-fields of events, denoted by $\mathcal{F}_n$ for $n \geq 0$. We know whether or not each of the events in $\mathcal{F}_n$ occurred by time n. We then write instead of (ii) above:

$$(ii) E[X_{n+1}|\mathcal{F}_n] = X_n \quad \text{for all} \quad n \geq 0 \ ,$$

where $\mathcal{F}_n$ is understood to be the history up to time n. With that notation, we assume the history is cumulative, starting at time 0. Then $\mathcal{F}_n$ can be understood to be shorthand for $\{Y_k : 0 \leq k \leq n\}$.

b. Conditional Expectation

In order to understand the definitions above, we need to understand conditional expectation. The basic concepts are reviewed in the first four sections of Chapter 3 in Ross. In particular, we need to know what $E[X|Y]$ means for random variables or random vectors X and Y. For this, see p. 106 of Ross.

By $E[X|Y]$, we mean a random variable. In particular, $E[X|Y] = E[X|Y = y]$ when $Y = y$. Thus $E[X|Y]$ can be regarded as a deterministic function of the random variable Y, which makes it itself be a random variable. Since (in the discrete case)

$$E[X] = \sum_y E[X|Y = y]P(Y = y) = E[E[X|Y]] \ ,$$

we have the fundamental relation

$$E[E[X|Y]] = E[X]$$

for all random variables X and Y.

As a consequence, for a martingale $\{X_n : n \geq 0\}$ with respect to $\{Y_n : n \geq 0\}$, we have

$$E[X_{n+1}] = E[E[X_{n+1}|Y_0, Y_1, \ldots, Y_n]] = E[X_n] \quad \text{for all} \quad n \geq 0 \ .$$

Thus, by mathematical induction, for a martingale $E[X_n] = E[X_0]$ for all $n \geq 1$. This last expected-value relation is a consequence of the martingale property, but it is not equivalent; the martingale property implies more than that.

c. Stopping Times.

An important concept associated with martingales is the notion of a stopping time.

Definition 0.2 *Let T be a random variable taking values in the nonnegative integers and let $\{Y_n : n \geq 0\}$ be a stochastic process. We say that T is a* ***stopping time*** *with respect to $\{Y_n : n \geq 1\}$ if the event $\{T \leq n\}$ is a function of $(Y_0, Y_1, \ldots, Y_n)$ for each $n \geq 0$ and $P(T < \infty) = 1$.*

Note that a stopping time is not only a random time (a random variable taking values in the nonnegative integers, when we work in discrete time), but it is a random time whose values depend on the history. We know whether or not $T \leq n$ by observing the history $(Y_0, Y_1, \ldots, Y_n)$. If we used the notation $\mathcal{F}_n$, then we would say that T is a stopping time if the event $\{T \leq n\}$ belongs to the sigma-field of events $\mathcal{F}_n$.

What is not a stopping time? We now give an example of a random time that is not a stopping time. First suppose T is the first time n such that $Y_1 + \cdots + Y_n = 4$; then T is a stopping time. On the other hand, suppose that T is the *last* time n among the integers $1 \leq n \leq 20$ such that $Y_1 + \cdots + Y_n = 4$; then T is **not a stopping time**. Clearly, the event $\{T \leq 7\}$ does not depend only on $(Y_0, Y_1, \ldots, Y_7)$. Instead, it depends upon $(Y_0, Y_1, \ldots, Y_{20})$.

d. The Optional Stopping Theorem (OST).

Stopping times play an important role with martingales because of the **optional stopping theorem (OST)** or Martingale Stopping Theorem. Stopping times enable us to compute the expected value of $E[X_T]$, where X_T is understood to mean X_n with the random index T. In particular, X_T is understood to be X_n when $T = n$. Hence,

$$E[X_T] = \sum_{n=0}^{\infty} E[X_n|T = n]P(T = n) \ .$$

Under special conditions, we will have the simple relation $E[X_T] = E[X(0)]$.

Here is a version of the OST with more stringent conditions than are actually required:

Theorem 0.1 *Suppose that $\{X_n : n \geq 0\}$ is a martingale with respect to $\{Y_n : n \geq 0\}$ and T is a stopping time with respect to $\{Y_n : n \geq 0\}$. Then $\{X_{T\wedge n} : n \geq 0\}$ is a martingale with respect to $\{Y_n : n \geq 0\}$, where $T \wedge n \equiv \min\{T, n\}$. If, in addition, either (i) T is bounded, i.e., $P(T < K) = 1$ for some K or (ii) $\{X_{T\wedge n} : n \geq 0\}$ is bounded, i.e., $P(|X_{T\wedge n}| < K) = 1$ for all n, then*

$$E[X_T] = E[X(0)] \ .$$

We use this last result in the Brownian motion exercises, as explained at the end.

Proof of the first part of Theorem 0.1. By the definition of a stopping time, the event $\{T = k\}$ is determined by $(Y_0, Y_1, \ldots, Y_k)$ and thus by $(Y_0, Y_1, \ldots, Y_n)$ for each k, $0 \leq k \leq n$.

Also, the event $\{T > n\}$, being the complement of the event $\{T \leq n\}$ is determined by $(Y_0, Y_1, \ldots, Y_n)$. Hence,

$$E[X(T \wedge n+1)|Y_0, \ldots, Y_n] = E[X(k \wedge n+1)|Y_0, \ldots, Y_n] = X(k) \quad \text{when} \quad T = k$$

for $0 \leq k \leq n$, and

$$E[X(T \wedge n+1)|Y_0, \ldots, Y_n] = E[X(n+1)|Y_0, \ldots, Y_n] = X(n) \quad \text{when} \quad T > n \ .$$

Combining these relations, we see that indeed,

$$E[X(T \wedge n+1)|Y_0, \ldots, Y_n] = X(T \wedge n) \ ,$$

as claimed. That is the key martingale property. Moreover,

$$E[|X(T \wedge n)|] = \sum_{k=0}^{n-1} E[|X(k)|]P(T = k) + E[|X(n)|]P(T \geq n) < \infty \ ,$$

using the martingale property for the process $\{X_n : n \geq 0\}$. ▪

2. Gambling

let $\{X_n : n \geq 0\}$ be a sequence of independent **gambles**. let $\{W_n : n \geq 0\}$ be a stochastic process, where W_n represents your **wealth** after gamble n. let $\{B_n : n \geq 0\}$ be a sequence of **bets**. We understand that B_n is the bet on gamble X_n. Your net gain from gamble n is $B_n X_n$. We require that your bet on gamble n can only depend on prior information, which includes the outcome of past gambles and the amounts of past bets. In other words, B_n is understood to be a function of $(W_0, X_1, \ldots, X_{n-1}, B_1, \ldots, B_{n-1})$ for each $n \geq 1$. Thus, if $(W_0 = w_0, X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, B_1 = b_1, \ldots, B_{n-1} = b_{n-1})$, then $B_n = b_n \equiv g_n(w_0, x_1, \ldots, x_{n-1}, b_1, \ldots, b_{n-1})$ where $g_n : \mathbb{R}^{2n-1} \to \mathbb{R}$ is a real-valued function of $2n-1$ real variables for each n. Your wealth at time n (after gamble n) is then

$$W_n = W_0 + \sum_{k=1}^{k=n} B_k X_k, \quad n \geq 1 \ .$$

For example, X_k may be a gamble based on a coin toss. In this gamble you win 1 for each dollar you bet if heads comes up, but you lose 1 for each dollar you bet if tails comes up. In other words, if you bet B_k, then you end up with $B_k X_k$, where $X_k = 1$ if heads shows up and $X_k = -1$ if tails shows up. The sequence of gambles could be a sequence of coin tosses, so that the sequence $\{X_n : n \geq 0\}$ is i.i.d. or, more generally, it could be a sequence of independent gambles, where the gamble changes with n. These gambles are **fair gambles** if $E[X_n] = 0$ for all n. Martingales help us analyze the consequence of fair gambles.

Theorem 0.2 *Consider the gambling scenario above, in which we have a sequence of independent gambles $\{X_n : n \geq 1\}$, where $E[|X_n|] < \infty$ and $E[X_n] = 0$ for all n. We require that your bet on gamble n can only depend on prior information, which includes the outcome of past gambles and the amounts of past bets. In other words, B_n is understood to be a function of $(W_0, X_1, \ldots, X_{n-1}, B_1, \ldots, B_{n-1})$. If $E[W_0] < \infty$ and there exists some constant K such that $B_n \leq K$ for all n, then the wealth stochastic process $\{W_n : n \geq 0\}$ is a martingale with respect to the history $\{W_0, B_1, X_1, \ldots, B_n, X_n : n \geq 1\}$; i.e.,*

$$(i) E[|W_n|] < \infty \quad \textit{for all} \quad n$$

and

$$(ii) E[W_{n+1}|W_0, B_1, X_1, \ldots, B_n, X_n] = W_n \quad \textit{for all} \quad n \geq 0$$

which implies that

$$E[W_n|W_0] = W_0 \quad \textit{and} \quad E[W_n] = E[W_0] \ .$$

Proof. Since the expected value of a sum of random variables is the sum of expected values,

$$
\begin{aligned}
E[|W_n|] &= E[|W_0 + \sum_{k=1}^{k=n} B_k X_k|] \\
&\leq E[|W_0|] + \sum_{k=1}^{k=n} E[|B_k X_k|] \\
&\leq E[|W_0|] + \sum_{k=1}^{k=n} E[|K X_k|] \\
&\leq E[|W_0|] + K \sum_{k=1}^{k=n} E[|X_k|] < \infty \ .
\end{aligned}
$$

Next,

$$
\begin{aligned}
E[W_{n+1}|W_0, B_1, X_1, \ldots, B_n, X_n] &= E[W_0 + \sum_{k=1}^{n+1} B_k X_k | W_0, B_1, X_1, \ldots, B_n, X_n] \\
&= W_0 + \sum_{k=1}^{k=n} B_k X_k + E[B_{n+1} X_{n+1} | W_0, B_1, X_1, \ldots, B_n, X_n] \\
&= W_n + E[B_{n+1} X_{n+1} | W_0, B_1, X_1, \ldots, B_n, X_n] \\
&= W_n + B_{n+1} E[X_{n+1} | W_0, B_1, X_1, \ldots, B_n, X_n] \\
&= W_n + B_{n+1} E[X_{n+1}] \\
&= W_n + B_{n+1} 0 = W_n \ .
\end{aligned}
$$

To explain the sequence of relations above, in the first line we simply write down the expression for W_{n+1}. In the second line, the random variables conditioned on those same random variables are the random variables themselves; i.e., we are using the relation $E[X|X] = X$. In the third line we observe that the initial terms combine to give W_n. In the fourth line, B_{n+1} factors out because we have assumed above that B_{n+1} is a function of the random variables we condition upon, namely, the history up to that time $(W_0, B_1, X_1, \ldots, B_n, X_n)$. The gambles are assumed to be independent, so X_{n+1} is independent of the history. Next, we have the expected value $E[X_{n+1}] = 0$. Hence we have the desired result. ∎

3. Popular Gambling Strategies

a. Doubling

The **doubling strategy** for a sequence of i.i.d. gambles with possible outcomes $+1$ and -1 is to start by betting some amount b and then the next time bet $2b$ if you lose. If you lose a second time, then you bet $4b$. If you keep doubling your bet each time you lose until you first win, then when you first win, you will be exactly b ahead. To see this, consider the following possible scenario.

Since you will eventually win sometime, you will eventually get b ahead. After you win, you start over, so that you get kb ahead after repeating this strategy k times. **What is the flaw?**

At any time, your expected wealth is the same as your initial wealth, by virtue of Theorem 0.2. We have $E[W_n] = E[W_0]$ for all n. You are ahead by b at the first time you win, but your may lose an arbitrarily large amount before you win. If we let T be the first time you win; i.e.,

$$T \equiv \min \{k \geq 1 : X_k = +1\} \ .$$

time period	0	1	2	3	4	5
bet		b	2b	4b	8b	16b
outcome		L	L	L	L	W
wealth	w	w - b	w - 3b	w - 7b	w - 15b	w + b

Table 1: A possible scenario for the doubling strategy. We start by betting b on the first gamble. In this scenario, we lose four bets and then win one. After these 5 bets, we are ahead by b.

Then, we are always ahead when we stop:

$$P(W_T = W_0 + b) = 1 \ .$$

As a consequence, here we have $E[W_T] = E[W_0] + b$, so that here we do **not** have $E[W_T] = E[W_0]$. But here the conditions of the OST are not satisfied. We do have $E[W_{T\wedge n}] = E[W_0]$ for all n, but we do not have $E[W_T] = E[W_0]$. To repeat, the extra conditions of the OST do not hold here. A careful examination of the OST explains what gambling strategies can achieve and what they cannot.

b. Constant Betting Until You Win.

Here is a second gambling strategy: The strategy of **constant betting until you are ahead** for a sequence of i.i.d. gambles with possible outcomes $+1$ and -1 is to start by betting some amount b and then continuing to bet b every time until the random (stopping) time at which the sum of the gambles is positive. In particular, we define T by

$$T \equiv \min\{k \geq 1 : X_1 + \cdots + X_k \geq 1\}$$

If $E[X_n] = 0$, then $P(T < \infty) = 1$; see Example 4.15 in Ross on pages 194-196. So, when T occurs, we are exactly b ahead. After winning b in this way, we repeat and win an arbitrarily large amount?

What is the flaw?

Again, we have

$$P(W_T = W_0 + b) = 1 \ ,$$

so that again $E[W_T] = E[W_0] + b$. The story is essentially the same as before: At any time n, your expected wealth is the same as your initial wealth. By virtue of Theorem 0.2. We have $E[W_n] = E[W_0]$ for all n. You are ahead by b at the first time you are ahead, but your may lose an arbitrarily large amount before you get ahead. If we use this new stopping time T, then we are always ahead when we stop:

$$P(W_T = W_0 + b) = 1 \ .$$

But again here we do **not** have $E[W_T] = E[W_0]$. Again, the conditions of the OST are not satisfied. We do have $E[W_{T\wedge n}] = E[W_0]$ for all n by the OST, but we do not have $E[W_T] = E[W_0]$. To repeat, the extra conditions of the OST do not hold here.

4. Martingales and Brownian motion

The above is intended to supplement the discussion of martingales associated with Brownian motion on p. 635 of Ross.

a. Exercise 10.17

Show that standard Brownian motion is a martingale. Since no history is mentioned, this means with respect to its own internal history. We need to show that (i) and (ii) hold in Definition 0.1. First,

$$|B(t)| \le 1 + B(t)^2 \ , \quad \text{so that} \quad E[|B(t)|] \le 1 + Var(B(t)) \le 1 + t < \infty \ .$$

Second, we need to establish the continuous analog of (ii) in Definition 0.1, namely,

$$E[B(t)|B(u), 0 \le u \le s < t] = B(s) \quad \text{for all} \quad 0 \le s < t \ .$$

To do so, we exploit independent increments, writing $B(t) = B(s) + (B(t) - B(s))$. then

$$\begin{aligned} E[B(t)|B(u), 0 \le u \le s < t] &= E[B(s) + B(t) - B(s)|B(u), 0 \le u \le s < t] \\ &= E[B(s)|B(u), 0 \le u \le s < t] + E[B(t) - B(s)|B(u), 0 \le u \le s < t] \\ &= B(s) + E[B(t) - B(s)] = B(s) \ . \end{aligned}$$

b. Exercise 10.5

We use stopping times and the optional stopping theorem to answer question 10.5. Let $T = T_a \wedge T_b$ be the minimum of the times to hit a and b. If $a < 0 < b$, then the continuous-time martingale $\{B(T \wedge t) : t \ge 0\}$ is bounded, so that the conditions in the OST are satisfied. Hence we can solve this problem by using the fact that

$$E[B(T)] = E[B(0)] = B(0) = 0 \ .$$

Hence,

$$P(T_{-1} < T_1) = 1/2$$

and

$$P(T_{-a} < T_b) = b/(a + b) \ .$$

Then

$$\begin{aligned} P(T_1 < T_{-1} < T_2) &= P(T_1 < T_{-1}) P(T_{-1} < T_2 | T_1 < T_{-1}) \\ &= P(T_1 < T_{-1}) P(T_{-2} < T_1) \\ &= (1/2)(1/3) = 1/6 \end{aligned}$$