IEOR 3106: Second Midterm Exam, Chapters 5-6, November 7, 2013

SOLUTIONS

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. You may keep the exam itself. Solutions will eventually be posted on line.

1. Copier Breakdown and Repair (35 **points)**

Three copier machines operate continuously and independently through time. They are maintained by a single repairman. Each copier functions for an exponentially distributed amount of time with mean 10 days before it breaks down. The repair times for each copier are exponential with mean 1 day, but the repairman can only work on one machine at a time. Assume that the machines are repaired in the order in which they fail.

(a) Suppose that all three copiers are initially working. What is the expected time until one of the copiers breaks down? (5 points)

Since the mean time until failure is 10 days for each copier, the failure rate for each copies is $\lambda = 1/10$ per day. The time until the first failure, say T_1, is the minimum of three i.i.d. exponential random variables, each with rate $\lambda = 1/10$. Thus the time until the first failure is also exponential with rate $3\lambda = 3/10$ and mean

$$E[T_1] = \frac{1}{3/10} = \frac{10}{3} \quad \text{days.}$$

(b) After the first copier breaks down, what is the probability that a second copier fails before the first one is repaired? (5 points)

Let X be the time until the next repair and let Y be the time until the next failure. By the lack of memory property, these are well defined at the time of the first failure, independent of the rest of the history. In particular, X is exponential with rate $\lambda_X = 1$ and Y is exponential with rate $\lambda_Y = 2/10$. Hence, we want

$$P(Y < X) = \frac{\lambda_Y}{\lambda_X + \lambda_Y} = \frac{2/10}{1 + (2/10)} = \frac{2}{12} = \frac{1}{6}.$$

(c) Let $X(t)$ be the number of copiers not working at time t. Multiple choice; pick the best answer(and explain): (5 points)

(i) The stochastic process $\{X(t) : t \geq 0\}$ is a Markov process.
(ii) The stochastic process $\{X(t) : t \geq 0\}$ is a birth-and-death process.
(iii) Both of the above.
(iv) None of the above.

The correct answer is (iii). A birth-and-death process IS a CTMC, which IS a Markov process.

(d) What is the long-run proportion of time that no copier is working? (10 points for correct model and analysis for parts (d) and (e), 5 points more for specific answers to questions)

Let $X(t)$ be the number of copiers not working at time t. (We could also have worked with the number of copiers working at time t.) The stochastic process $\{X(t) : t \geq 0\}$ is a CTMC, specifically a BD process. For specifying the model, it is good to draw a rate diagram, as done in §4 of the CTMC notes. Hence, the long run proportion of time that there are j failed copiers, $0 \leq j \leq 3$, is α_j, where $\alpha Q = 0$ and, because of the BD property,

$$\alpha_j = \frac{r_j}{\sum_k r_k}, \quad j \geq 0, \quad \text{where} \quad r_j \equiv \frac{\lambda_0 \lambda_1 \times \cdots \lambda_{j-1}}{\mu_1 \mu_2 \times \cdots \mu_j} \quad \text{and} \quad r_0 \equiv 1.$$

Here the birth rates are

$$\lambda_0 = \frac{3}{10}, \quad \lambda_1 = \frac{2}{10}, \quad \lambda_2 = \frac{1}{10},$$

and the death rates are

$$\mu_1 = 1, \quad \mu_2 = 1, \quad \mu_3 = 1.$$

Hence,

$$r_0 = 1, \quad r_1 = \frac{3}{10}, \quad r_2 = \frac{6}{100} \quad \text{and} \quad r_3 = \frac{6}{1000}$$

so that

$$\alpha_0 = \frac{1000}{1366}, \quad \alpha_1 = \frac{300}{1366}, \quad \alpha_2 = \frac{60}{1366} \quad \text{and} \quad \alpha_3 = \frac{6}{1366}$$

Finally, the long-run proportion of time that no copier is working

$$\alpha_3 = \frac{6}{1366}.$$

(e) What is the long-run proportion of time that the repairman is busy doing repair work on these copiers? (5 points for specific question)

Using part (c),

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 - \alpha_0 = 1 - \frac{1000}{1366} = \frac{366}{1366}$$

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2. Fishing (35 points, 5 points each part)

Suppose that a fisherman catches fish at random times, according to a Poisson process with rate 4 fish per hour. Suppose that each fish is either a grouper or a snapper, with the probability of being a grouper being 1/4 (independent of the history up to that time). Let W_g and W_s be the random weights of each grouper and snapper, respectively, (also independent of the history), with means and standard deviations:

$$E[W_g] = 100, \quad SD[W_g] = 20, \quad E[W_s] = 20, \quad \text{and} \quad SD[W_s] = 10,$$

measured in pounds.

(a) What are the mean and variance of the time until the fisherman catches his fourth fish?

Let T be the time until catching the fourth fish. Since the times between catching successive fish are i.i.d exponential random variables with mean $1/\lambda = 1/4$ hour,

$$E[T] = \frac{4}{4} = 1 \quad \text{hour} \quad \text{and} \quad Var(T) = \frac{4}{4^2} = \frac{1}{4}.$$

(b) What is the probability that the fisherman catches exactly 6 fish in a given 2-hour period?

Let $N(t)$ be the number of fish caught up to time t. Since $N(t)$ has a Poisson distribution and $E[N(2)] = 4 \times 2 = 8$,

$$P(N(t+2) - N(t) = 6) = \frac{e^{-8}(8)^6}{6!}$$

(c) What is the conditional probability that the fisherman catches exactly 6 fish in a given 2-hour period, given that he catches no fish in the previous two hours?

Again, let $N(t)$ be the number of fish caught up to time t. Since a Poisson process has independent increments,

$$P(N(t+2) - N(t) = 6 | N(t) - N(t-2) = 0) = P(N(t+2) - N(t) = 6) = \frac{e^{-8}(8)^6}{6!},$$

the same answer as in part (b).

(d) What is the probability that the fisherman catches exactly 4 grouper in a given 2-hour period (along with an unspecified number of snapper)?

Let $N_G(t)$ and $N_S(t)$ be the number of grouper and snapper, respectively, caught up to time t. By independent thinning of a Poisson process, the stochastic processes $\{N_G(t) : t \geq 0\}$ and $\{N_S(t) : t \geq 0\}$ are independent Poisson Processes with rates $\lambda_G = \lambda/4 = 1$ and $\lambda_S = 3\lambda/4 = 3$. Hence,

$$P(N_G(t+2) - N_G(t) = 4) = \frac{e^{-2}2^4}{4!}$$

(e) What is the probability that the fisherman catches exactly 4 grouper and 5 snapper in a given 2-hour period?

Since the stochastic processes $\{N_G(t) : t \geq 0\}$ and $\{N_S(t) : t \geq 0\}$ are independent Poisson Processes with rates $\lambda_G = \lambda/4 = 1$ and $\lambda_S = 3\lambda/4 = 3$, respectively,

$$\begin{aligned} P(N_G(t+2) - N_G(t) = 4, N_S(t+2) - N_S(t) = 5) &= P(N_G(t+2) - N_G(t) = 4) P(N_S(t+2) - N_S(t) = 5) \\ &= \left(\frac{e^{-2}2^4}{4!}\right)\left(\frac{e^{-6}6^5}{5!}\right) \end{aligned}$$

(f) What are the mean and variance of the total weight of all fish caught by the fisherman in a given two-hour period?

The total weight of all fish caught up to time t, denoted by $W(t)$ is a compound Poisson process, i.e.,

$$W(t) = \sum_{i=1}^{N(t)} X_i,$$

where $X_i : i \geq 1\}$ is a sequence of i.i.d. random variables distributed as X, where

$$E[X] = \frac{E[W_g]}{4} + \frac{3E[W_s]}{4} = \frac{100}{4} + \frac{3(20)}{4} = 25 + 15 = 40$$

and

$$E[X^2] = \frac{E[W_g^2]}{4} + \frac{3E[W_s^2]}{4} = \frac{100^2 + 20^2}{4} + \frac{3(20^2 + 10^2)}{4} = 2600 + 375 = 2975$$

Then

$$E[W(t)] = E[N(t)]E[X] = \lambda t E[X] \quad \text{and} \quad Var(W(t)) = E[N(t)]E[X^2]\lambda t E[X^2]$$

so that

$$E[W(2)] = 8E[X] = 320 \quad \text{and} \quad Var(W(2)) = 8E[X^2] = 23,800.$$

(g) What is the approximate probability that the total weight of all fish caught by the fisherman in a given two-hour period exceeds 400 pounds? Multiple choice; pick the best answer (and explain; justify your answer):

(i) 1.00
(ii) 0.50
(iii) 0.30
(iv) 0.03
(v) 0.00

Since the stochastic process $\{W(t) : t \geq 0\}$ is a compound Poisson process, it has stationary and independent increments. Hence, once can invoke the central limit theorem and apply a normal approximation.

$$\begin{aligned}
P(W(2) > 400) &= P\left(\frac{W(2) - E[W(2)]}{SD(W(2))} > \frac{400 - E[W(2)]}{SD(W(2))}\right) \\
&\approx P\left(N(0,1) > \frac{400 - E[W(2)]}{SD(W(2))}\right) \\
&\approx P\left(N(0,1) > \frac{400 - 320}{\sqrt{23,800}}\right) \\
&\approx P(N(0,1) > 0.5) \approx 0.3
\end{aligned}$$

using the table of the normal distribution. A rough estimate of the square root is enough:

$$\sqrt{23,800} = 154 \quad \text{or} \quad \approx \sqrt{22,500} = 150 \quad \text{or} \quad \approx \sqrt{25,600} = 160$$

giving $\approx P(N(0,1) > 0.5) = 0.3$, as in (iii). The other choices are not close. A more accurate value is $\approx P(N(0,1) > 0.52)$, but that was not wanted.

It is important to be able to quickly see that $23,800 = 238 \times 100$, so that $\sqrt{23,800} = 10\sqrt{238} \approx 150$.

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3. Cars in a Highway Segment (30 points, 5 points each part)

Suppose that cars enter a (one-way) highway segment at an increasing rate over some interval of time. Specifically, suppose that cars enter the highway segment according to a nonhomogeneous Poisson process with rate $\lambda(t) = 18t$ per minute at time t, starting at time 0. Assume that different cars do not interact. Suppose that the time each car remains in the highway segment is a random variable uniformly distributed on the interval $[2,4]$ minutes. Suppose that these random times for different cars are mutually independent. Let $A(t)$ be the number of cars to enter the highway segment during $[0,t]$ and let $X(t)$ be the number of cars in the highway segment at time t.

(a) Give an (exact) expression for $E[A(10)]$.

$$E[A(t)] = \int_0^t \lambda(s)\, ds = \int_0^t 18s\, ds = 9t^2,$$

so that

$$E[A(10)] = 9(10)^2 = 900$$

(b) Give an approximate expression for $P(A(10) > 950)$.

Since $A(10)$ has a Poisson distribution and $E[A(10]$ is large, we can use a normal approximation for the Poisson distribution:

$$P(A(10) > 950) \approx P\left(N(0,1) > \frac{950-900}{30}\right) = P(N(0,1) > 1.67) \approx 0.0475$$

using the table of the normal distribution.

(c) Give an (exact) expression for $P(A(2) = 40|A(1) = 20)$.

First, $A(2) = A(1) + (A(2) - A(1))$. Second, because of independent increments,

$$P(A(2) = 40|A(1) = 20) = P(A(2) - A(1) = 20|A(1) = 20) = P(A(2) - A(1) = 20),$$

which has a Poisson distribution with mean m, i.e.,

$$P(A(2) - A(1) = 20) = \frac{e^{-m}m^{20}}{20!},$$

where

$$m \equiv E[A(2) - A(1)] = \int_1^2 18s\, ds = 36 - 9 = 27$$

(d) Give an (exact) expression for the covariance $Cov[A(10)-A(0), A(30)-A(20)]$. (Recall that $Cov(X,Y) = E[XY] - E[X]E[Y]$.)

Since the NHPP has independent increments, these two increments are independent. Two independent random variables are always uncorrelated. Thus, we have

$$Cov[A(10) - A(0), A(30) - A(20)] = 0$$

(e) Give an (exact) expression for $E[X(10)]$.

The random variable $X(t)$ represents the number of busy servers in an infinite-server queue with an NHPP arrival process. Thus, as given on the formula sheet,

$$m(t) \equiv E[X(t)] = \int_0^t \lambda(t-s)P(S > s)\, ds,$$

where S is a service time (the time that each car remains in the segment). (3 points to here.) But the distribution of S should be specified. It satisfies

$$P(S > s) = 1 \quad \text{for} \quad s \le 2, \quad P(S > s) = 0 \quad \text{for} \quad s \ge 4,$$

and

$$P(S > s) = (4-s)/2 \quad \text{for} \quad 2 \le s \le 4.$$

since S is uniform over $[2, 4]$. (4 points to here.) One more point for detailed calculation:

However, we can carry out the integration. We can break the integral into two parts:

$$\begin{aligned}
m(10) &\equiv E[X(10)] = \int_0^{10} \lambda(10-s)P(S > s)\, ds, \\
&= \int_0^2 \lambda(10-s)\, ds + \int_2^4 \lambda(10-s)(4-s)/2\, ds, \\
&= \int_8^{10} \lambda(s)\, ds + \int_2^4 \lambda(10-s)(4-s)/2\, ds, \\
&= \int_8^{10} 18s\, ds + \int_2^4 (180-18s)(4-s)/2\, ds, \\
&= \int_8^{10} 18s\, ds + \int_2^4 (360-126s+9s^2)\, ds, \\
&= 324 + (720 - (63 \times 12) + (3 \times 56) = 324 + 132 = 456
\end{aligned}$$

(f) Multiple choice; pick the best answer(and explain):

(i) The stochastic process $\{X(t) : t \ge 0\}$ is a nonhomogeneous Poisson process.
(ii) The stochastic process $\{X(t) : t \ge 0\}$ is a Markov process.
(iii) Both of the above.
(iv) None of the above.

The random variable $X(t)$ has a Poisson distribution, but the stochastic process $\{X(t) : t \geq 0\}$ is *not* an NHPP and *not* a Markov process. The correct answer is (iv).
